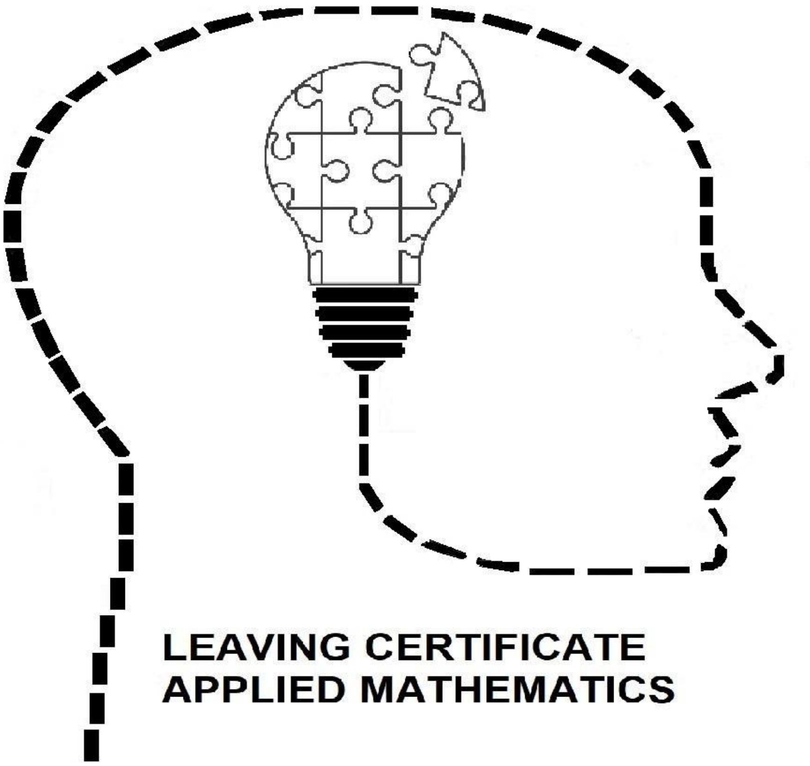
PDST
Professional Development Service for Teachers | An tSeirbhís um Fhorbairt Ghairmiúil do Mhúinteoirí

An Roinn Oideachais
Department of Education



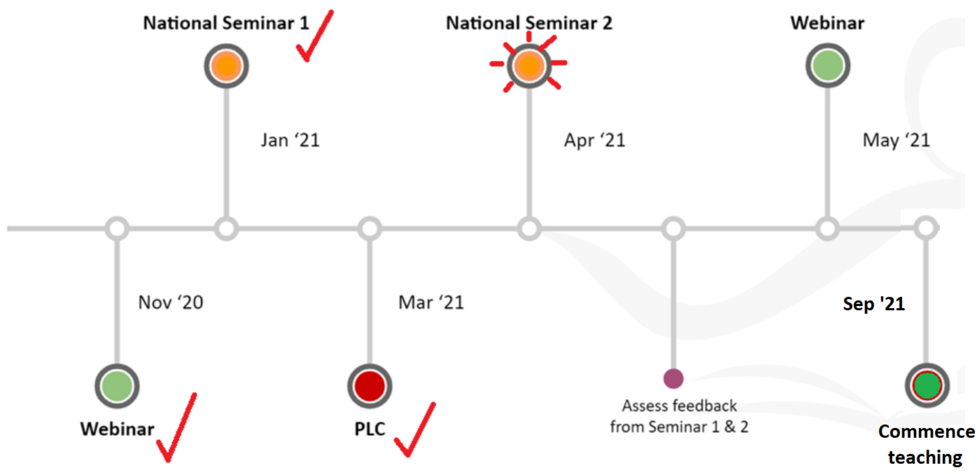
National Seminar 2
2021
Booklet

Approaches to Mathematical Modelling in the Classroom

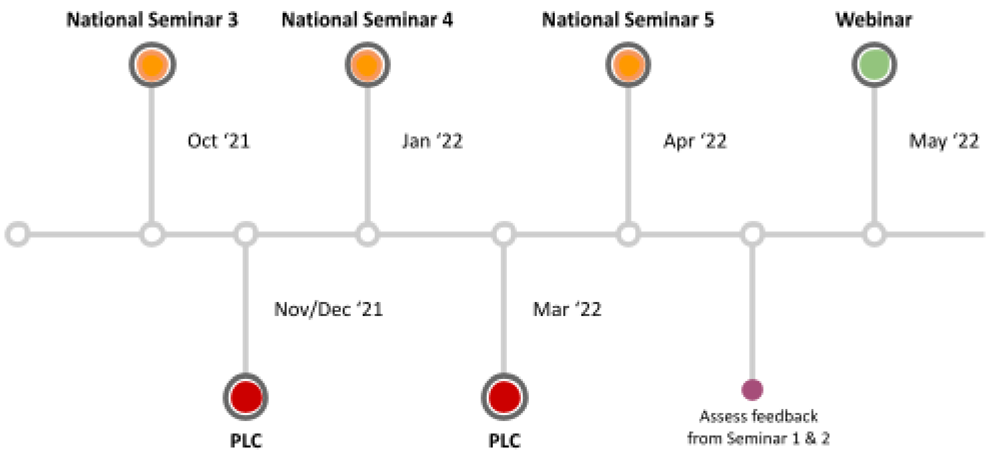
Introduction

Overview of Professional Development

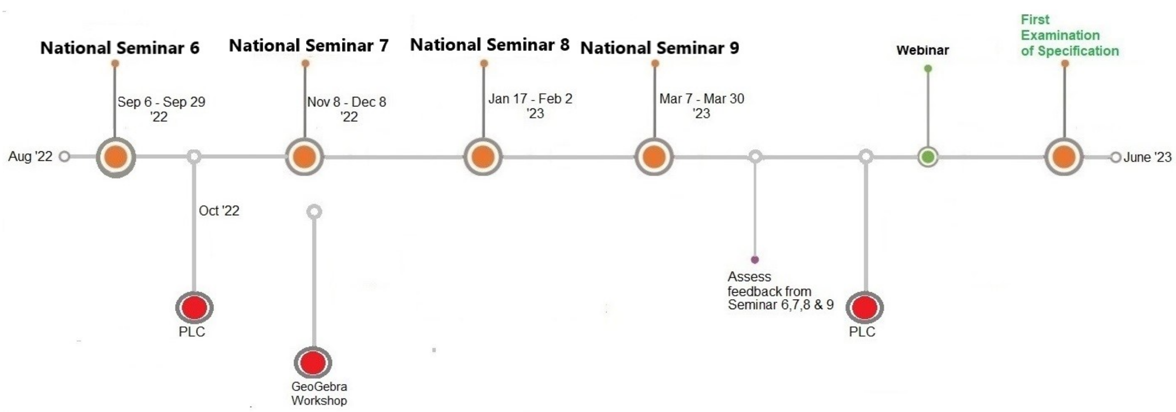
Year 1 Nov 2020 - June 2021



Year 2 Sep 2021 - June 2022



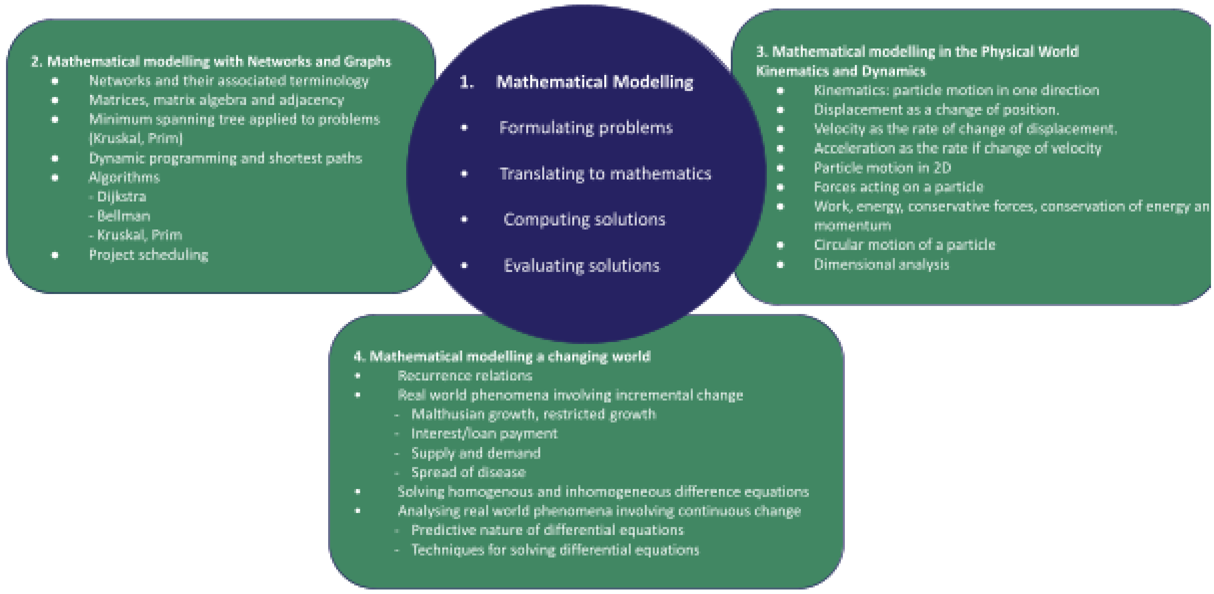
Year 3 Sep 2022 - June 2023



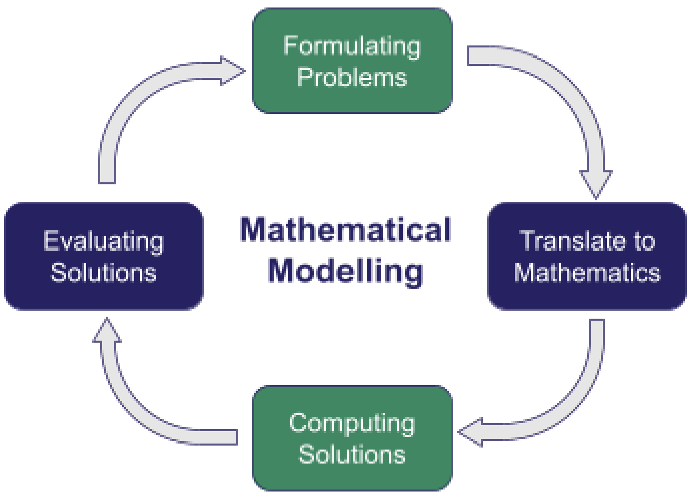
Key Messages

1. Core to the specification is a non-linear approach empowered by the use of rich pedagogy which promotes the making of connections between various Applied Mathematics learning outcomes.
2. Strand 1 of the specification is a unifying strand and emphasises the importance of utilising modelling across all learning outcomes.
3. Applied Mathematics is rooted in authentic problems as a context for learning about the application of Mathematics to design solutions for real-world problems and to develop problem solving skills applicable to a variety of disciplines.

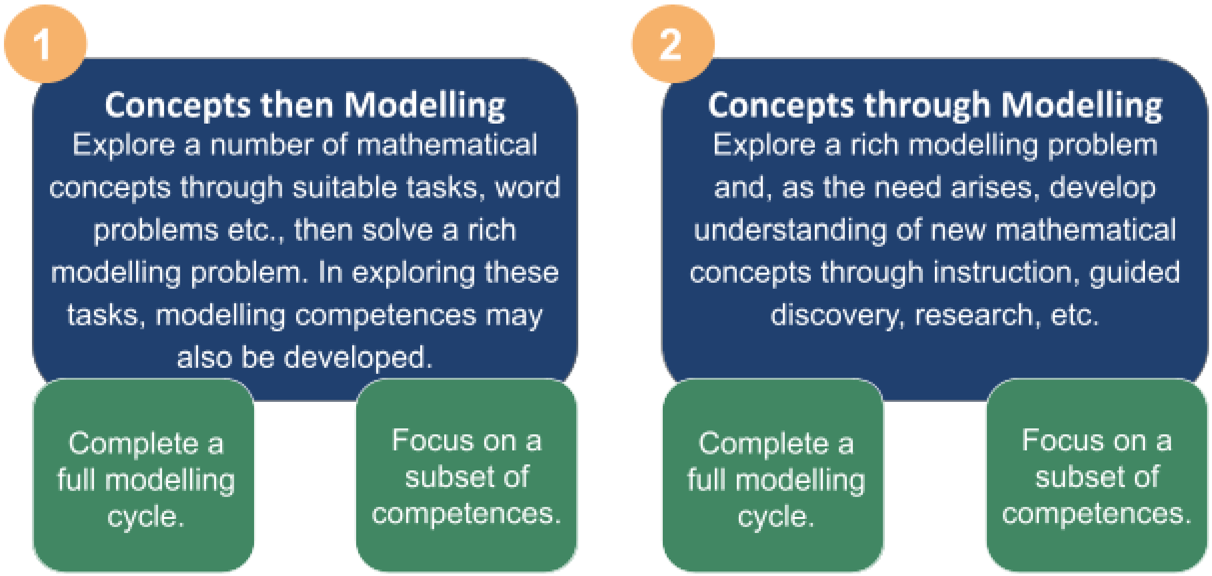
Leaving Cert Applied Mathematics Strands and Strand Units



What are the different stages in mathematical modelling?



Approaches to Mathematical Modelling in the Classroom



Curriculum 'Glance Cards[1]

A fundamental principle of the curriculum is that students' current understanding and knowledge should form the basis for new learning. The Applied Mathematics specification is designed to be taught in a non-linear fashion.

These curriculum "glance cards" were designed to provide a one-page overview of the content objectives in each strand unit.

It is not intended that these glance cards replace the specification documents but that rather they will provide an immediate snapshot of how particular concepts are developed for students throughout the two year cycle.

Teachers may find this useful when they are engaging in continuing professional development, or when they are planning for teaching and learning. However, it is essential that teachers consult the specification document when engaging in planning as the content objectives are expanded upon in the context of all of the Strands.

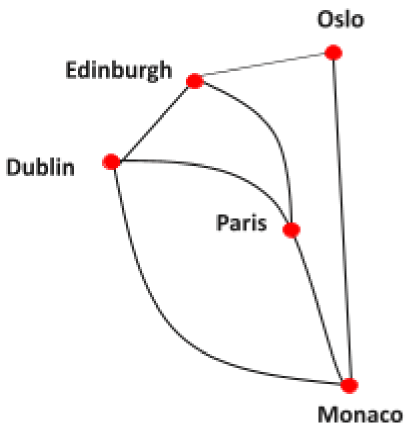
[1] Copy of Glance card in appendix

Session 1

Authentic Problem - Online Marketing

A company wishes to engage in an online marketing campaign. The company has limited resources to invest in such a campaign.
How will they create this marketing campaign?

Can you represent the connections between these cities in a more ordered fashion?



	OSLO	EDINBURGH	DUBLIN	PARIS	MONACO
OSLO					
EDINBURGH					
DUBLIN					
PARIS					
MONACO					

What defines a mathematical matrix?

A matrix is a rectangular array of numbers arranged in rows and columns, commonly written in [ ] square brackets.

A matrix with *m* rows and *n* columns is called an *m* X *n* matrix, *m* and *n* are called the matrix dimensions.

$$\begin{array}{c|c} & \text{COLUMN 1} \quad \text{COLUMN 2} \quad \text{COLUMN 3} \quad \text{COLUMN 4} \\ \hline \begin{array}{c}\text{Row 1}\\ \text{Row 2}\\ \text{Row 3}\\ \text{Row 4}\\ \text{Row 5}\end{array} & \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix} \end{array}$$

5 x 4

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

2 x 3

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

4 x 4

Matrix Multiplication - Method and Purpose

Matrix multiplication is only valid if the number of columns of the first matrix are equal to the number of rows of the second matrix.

The resulting matrix will have the number of rows of the first matrix and the number of columns of the second matrix.

$$m \times \cancel{n} \;\; \times \;\; \cancel{n} \times k \; = \; m \times k$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$4 \times 4 \quad \mathbf{x} \quad 4 \times 1 \; = \; 4 \times 1$$

4 columns in first matrix

4 rows in second matrix

Squaring a 3x3 matrix

$$\begin{bmatrix} \boxed{3 \;\; 1 \;\; 6} \\ 4 \;\; 2 \;\; 0 \\ 7 \;\; 8 \;\; 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 4 & 2 & 0 \\ 7 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 55 & R1C2 & R1C3 \\ R2C1 & R2C2 & R2C3 \\ R3C1 & R3C2 & R3C3 \end{bmatrix} = \begin{bmatrix} 55 & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

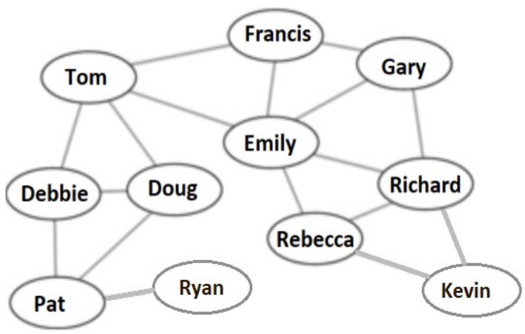
(3x3) + (1x4) + (6x7) = 55

Application of Adjacency Matrix

Consider the adjacency list below and complete an appropriate network graph.

<u>Adjacency List</u>
This is a list of nodes that are adjacent to each other.
1: 2
2: 1, 3, 4
3: 2, 4
4: 2, 3

Returning to our Problem: Who is the most connected person in this network?



	Tom	Debbie	Pat	Ryan	Doug	Emily	Francis	Rebecca	Richard	Gary	Kevin	
Tom	0	1	0	0	1	1	1	0	0	0	0	4
Debbie	1	0	1	0	1	0	0	0	0	0	0	3
Pat	0	1	0	1	1	0	0	0	0	0	0	3
Ryan	0	0	1	0	0	0	0	0	0	0	0	1
Doug	1	1	1	0	0	0	0	0	0	0	0	3
Emily	1	0	0	0	0	0	1	1	1	1	0	5
Francis	1	0	0	0	0	1	0	0	0	1	0	3
Rebecca	0	0	0	0	0	1	0	0	1	0	1	3
Richard	0	0	0	0	0	1	0	1	0	1	1	4
Gary	0	0	0	0	0	1	1	0	1	0	0	3
Kevin	0	0	0	0	0	0	0	1	1	0	0	2

Reflection on Teaching and Learning: Session 1

What teaching and learning strategy did I find the most useful this morning?

How can I suitably apply the various teaching and learning strategies demonstrated this morning?

What critical skills can I help develop in my students?

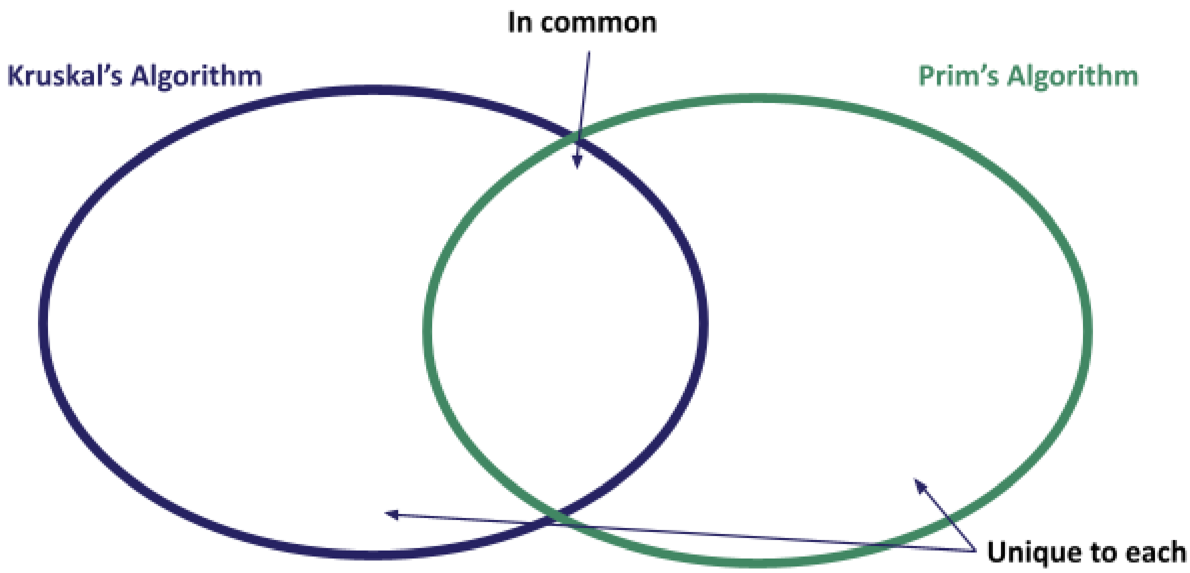
Session 2

Networks and Graph Theory: Algorithms

Graph Theory is a branch of Mathematics concerned with networks of points connected by lines.

STUDENTS LEARN ABOUT	STUDENTS SHOULD BE ABLE TO
‣ routing problems ‣ allocation of resources ‣ equipment replacement and maintenance	‣ apply Dijkstra's algorithm to find the shortest paths in a weighted undirected and directed network ‣ evaluate different techniques for solving shortest-route problems
Algorithms: ‣ Dijkstra ‣ Bellman ‣ Kruskal, Prim	‣ use algorithms to solve problems ‣ **distinguish between those algorithms which are greedy and those which use dynamic programming** ‣ **justify the use of algorithms in terms of correctness and their ability to yield an optimal solution**

What do we remember about algorithms from National Seminar 1?



Reflection: Connecting with Prior Knowledge

What are the benefits for students in connecting with previously learnt material?

How could we support students to be more personally effective in making connections between their learning?

What other strategies could be used to scaffold and build knowledge between concepts?

Recap of algorithms to date:

Kruskal's Algorithm:

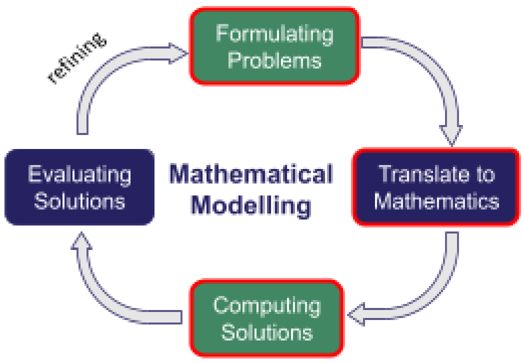
1. To begin, select the edge of least weight.
2. Find the next edge of least weight. If it would form a cycle with the edges already selected, don't choose it. If not then add it to the MST.
3. If there is a choice of equal edges, it has no effect which you choose first.
4. Repeat step 2 until all vertices are connected.

Prim's Algorithm:

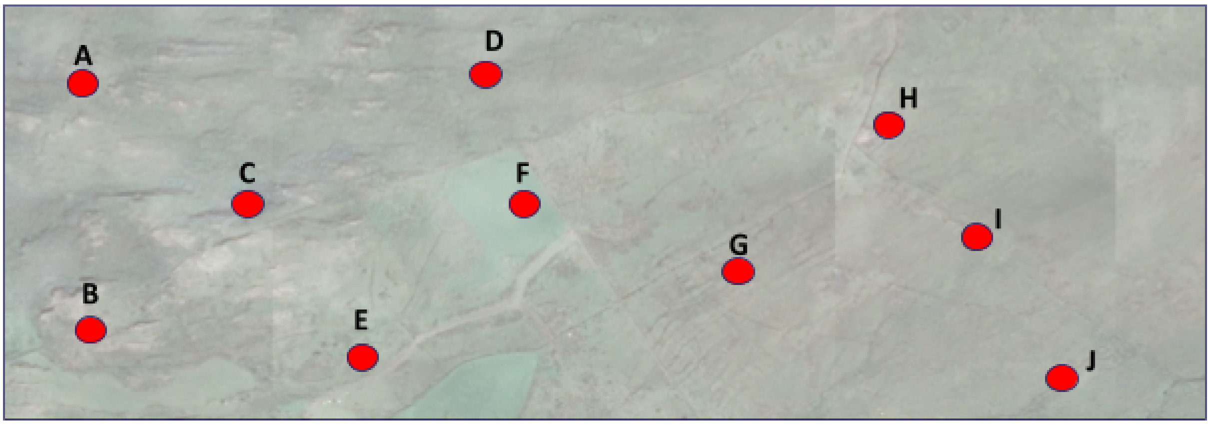
1. To begin, pick any vertex/node (unless a predetermined one is indicated).
2. Find all the edges that connect the tree to new nodes, select the minimum and add it to the tree, ensuring to avoid cycles.
3. Keep repeating step 2 until we get a minimum spanning tree with all nodes connected and cycles avoided.

It can be helpful to write a visited list to keep track of nodes that are already in the minimum spanning tree.

Applications of Previous Algorithms



A wildlife conservation park has a number of warden stations located within the park. The local authority closes the park to the public during the winter season. There are a number of roads within the park that connect the stations so that they are accessible to staff.
Calculate the minimum cost of ensuring road access to all the warden stations so that they are all connected, based on the cost per km to keep the roads maintained during the winter season.
What information is required to formulate and solve this problem?



	A	B	C	D	E	F	G	H	I	J
A	-	-	32	-	-	35	-	-	-	-
B	-	-	20	-	15	-	-	-	-	-
C	32	20	-	-	50	14	-	-	-	-
D	-	-	-	-	-	12	-	19	-	-
E	-	15	50	-	-	17	24	-	-	-
F	35	-	14	12	17	-	-	25	18	-
G	-	-	-	-	24	-	-	-	20	9
H	-	-	-	19	-	25	-	-	21	-
I	-	-	-	-	-	18	20	21	-	10
J	-	-	-	-	-	-	9	-	10	-

Each distance is measured in *km*. Cost = €80 per km

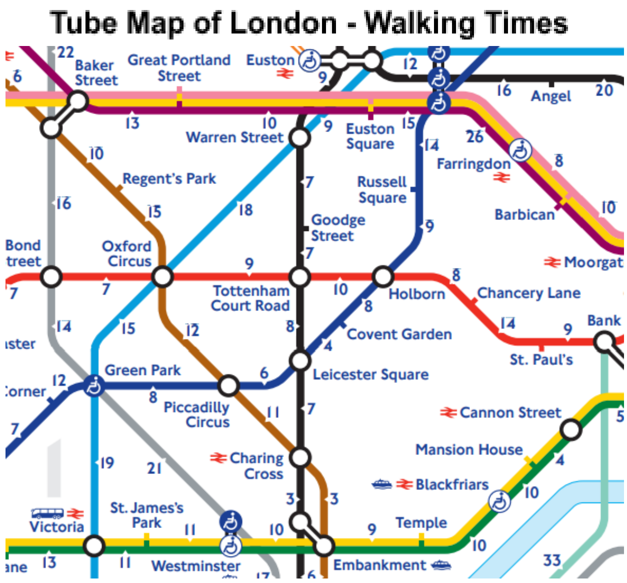
Task: Create the road network for the wildlife park from the table and determine the minimum total cost to ensure road access to all warden stations.

Authentic Problem - “Will we walk it?”

A 2015 study reported that sometimes it is quicker to walk between locations in London rather than take the underground train. This led to the Transport of London (TfL) to create the first official walking map of the tube. A portion of the map is shown.

Determine whether it is quicker to walk between Victoria and Holborn stations following the given tube walking routes or to use the underground train. Explain your answer.

Source: http://content.tfl.gov.uk/walking-tube-map.pdf



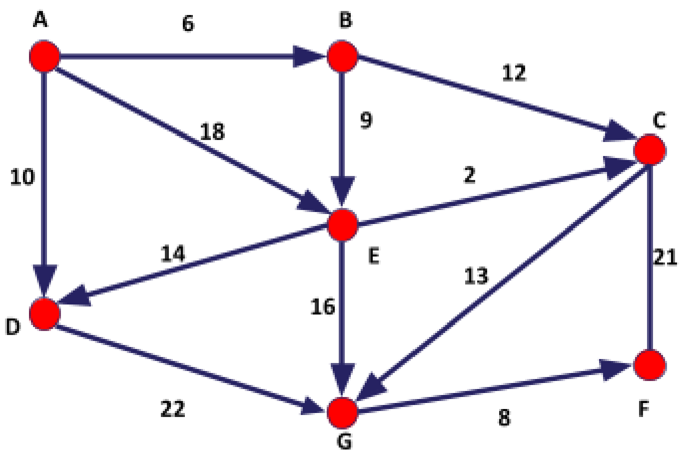
Quickest Pizza Delivery Route

Kevin has ordered a pizza from Little Nero’s Pizza (A). The pizza must be delivered in less than 40 minutes or else the pizza is free. The map shows the time taken to travel each street (in minutes) and the arrows represent the flow of traffic on a street.

(a) Find the route that the delivery person needs to choose so that they reach Kevin’s house (F) in the quickest time.

(b) Explain how you determined the quickest route.

Little Nero's Pizza



Kevin’s House

Describe the Approach

Create a formal step-by-step approach to find the shortest/quickest/cheapest path between two points in a network using suitable terminology.

Dijkstra's Algorithm

Used for finding the shortest, cheapest or quickest route between two vertices.

1. Label the start vertex as 0. Make this number a permanent label.
2. Record a working value at every vertex that is directly connected to the vertex that has just been made a permanent label.
3. Select the smallest working value of the vertices that do not have a permanent label. Make this a permanent label. If two vertices have the same smallest working value then choose either.
4. Repeat steps 2 and 3 until the destination vertex receives a permanent label.
5. To find the shortest path, trace back from the destination vertex to the start vertex.

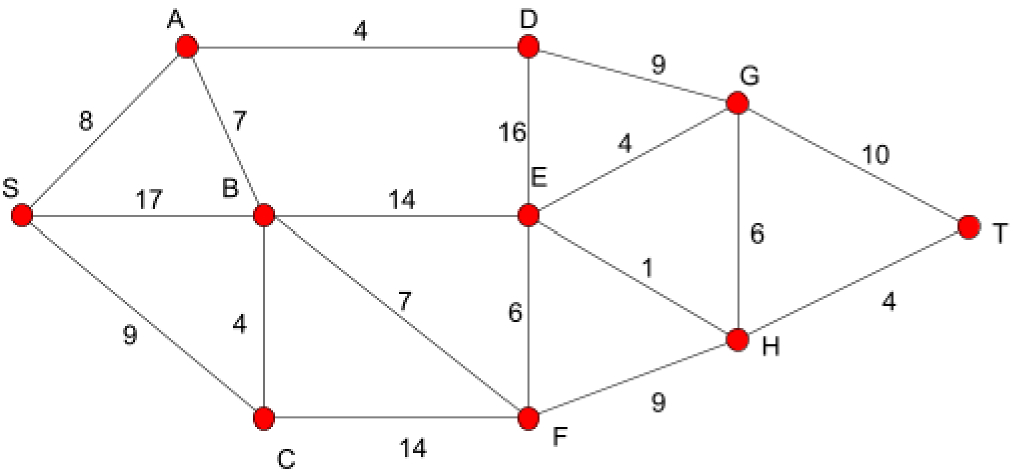
Note: If there are multiple start points, apply Dijkstra's algorithm from the end-point until you have reached each of the starting points.

Applying Dijkstra's Algorithm

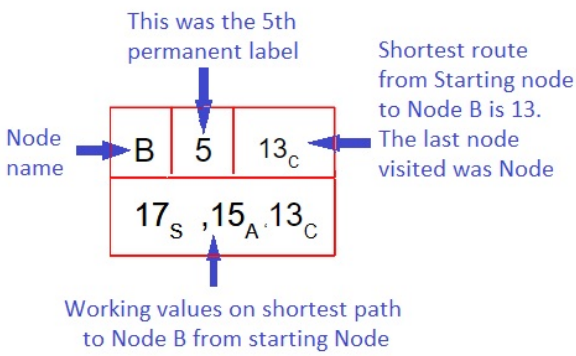
The graph below represents a network of roads with each weight representing distance in *km*. Liz wishes to travel from S to T.

(a) Use Dijkstra's Algorithm to find the shortest route from S to T. State the length of your route.

(b) On a particular day, Liz must include F in her route. Find the new shortest path and state its length.



DIJKSTRA KEY



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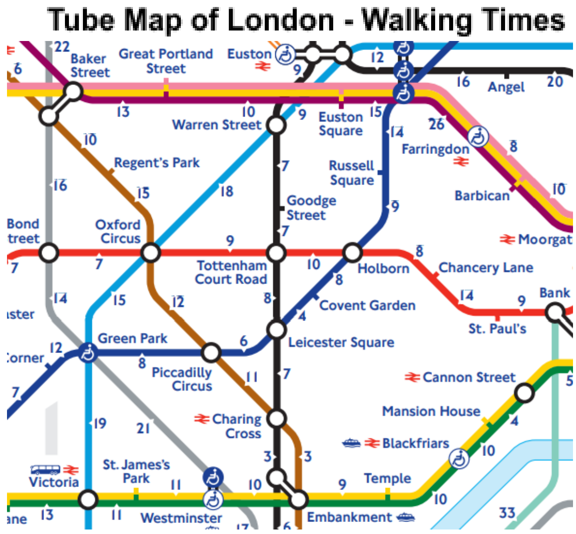
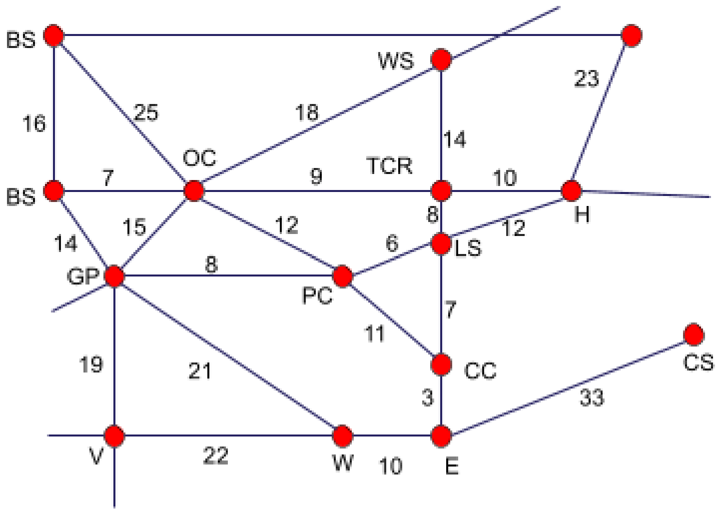
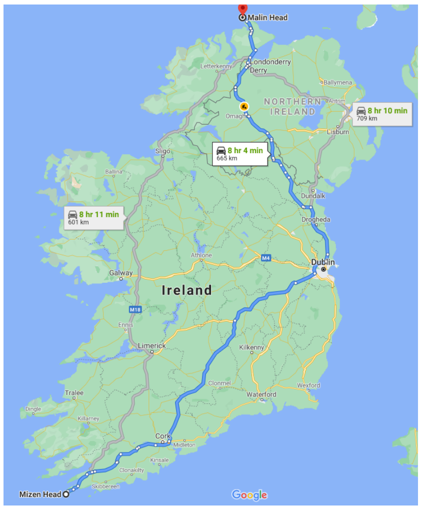
Determine whether it is quicker to walk between Victoria and Holborn following the given tube walking routes or to use the underground train.
Explain your answer.



Translate to Mathematics



Applications of Dijkstra's Algorithm

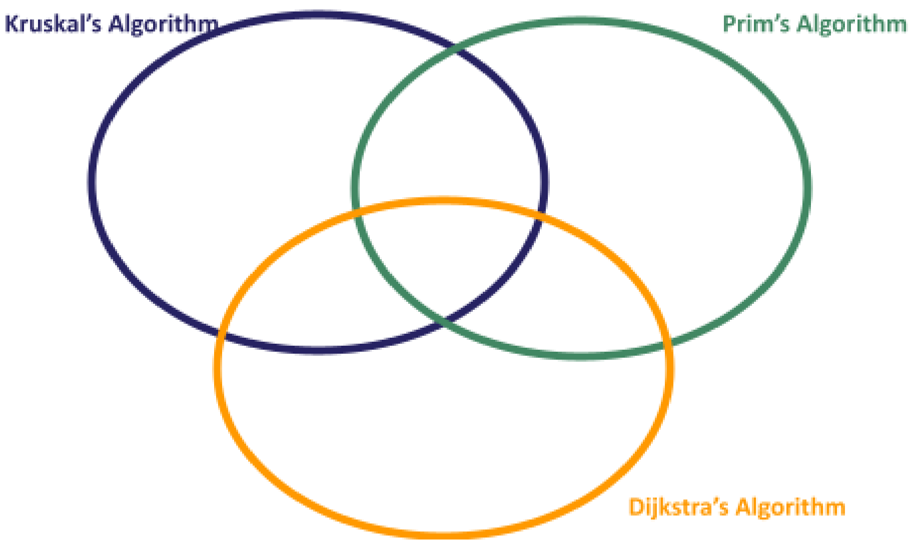


- Calculating the shortest/quickest path in Google Maps.
- Determining the line of least bandwidth in telephone networks.
- Friend suggestions in social networks (Facebook etc.) by analysing mutual friend connections.
- Automated delivery drones/robots determining the shortest path between their source and destination.
- Travel agents creating flight agendas for clients. The agent can calculate the earliest arrival time for a destination given an origin airport and start time.

Reflection on Algorithms

Analyse the three algorithms visited to date and discuss/comment on each using the following as guides:

- Purpose
- Teaching and learning
- Suitability to real life applications
- Other comments



Starts with a vertex

Starts with an edge

Follows a step-by-step method

More suitable for dense graphs

Can be refined to improve the model

Could be integrated with other strands

Creates a MST

Identifies a shortest path

Follows a formal algorithm

Can be used in Mathematical Modelling

Used to solve authentic problems

Other

Other

Reflection on Teaching and Learning: Session 2

Consider how the teaching and learning strategies used in this session support the integration of mathematical modelling in students' understanding of algorithms?

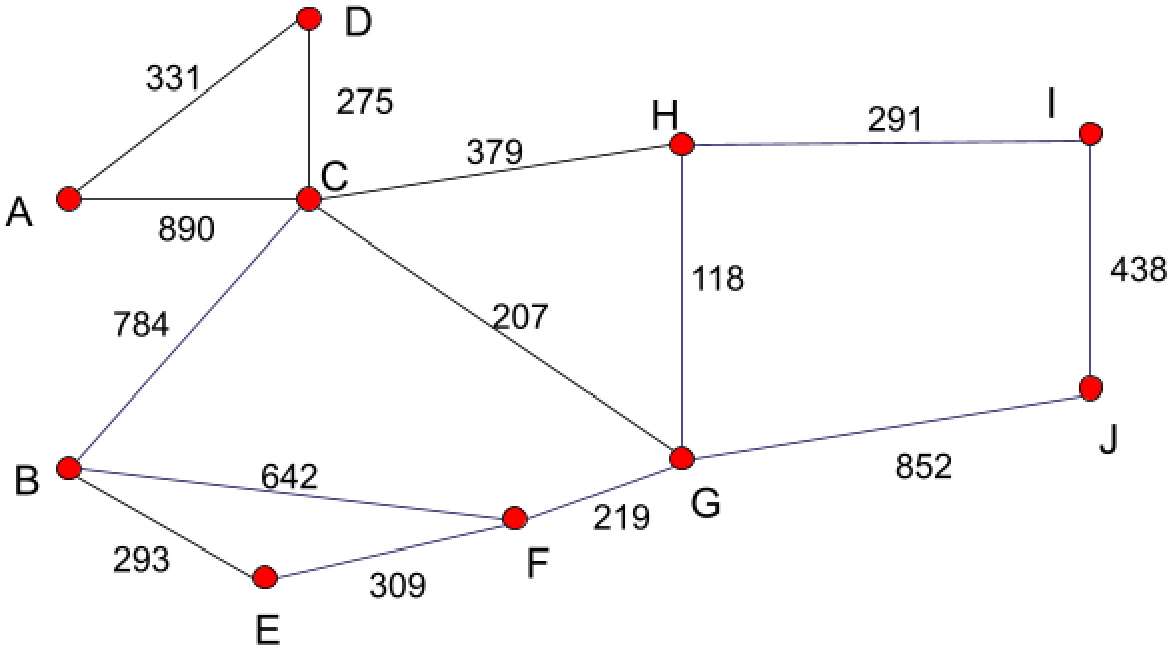
What student modelling key skills could be developed using the teaching and learning approaches demonstrated in this session?

Consider how the learning outcomes of Strand 1 were necessarily and purposefully integrated into our teaching and learning of algorithms from Strand 2.

Session 3

Authentic Problem - Modelling Response Times

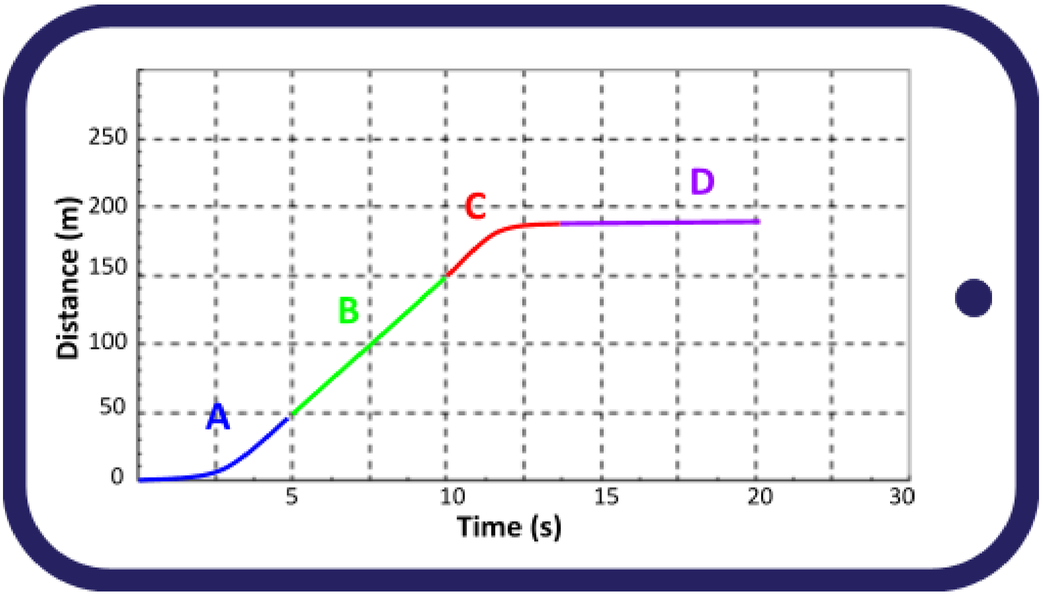
The network below represents the distances, in metres, on a road network. An accident has occurred at J and emergency services are called to the scene. An ambulance starts from A and a Garda motorbike starts from B at the same time with both vehicles trying to reach J as quickly as possible.
Which vehicle reaches J first?



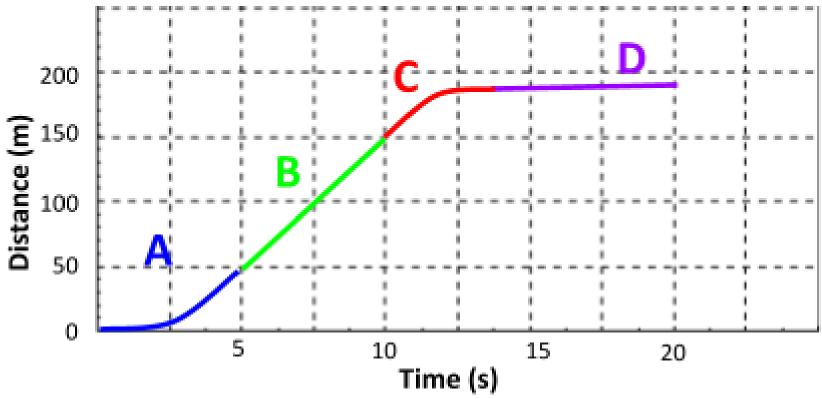
Describing the Ambulance's Motion

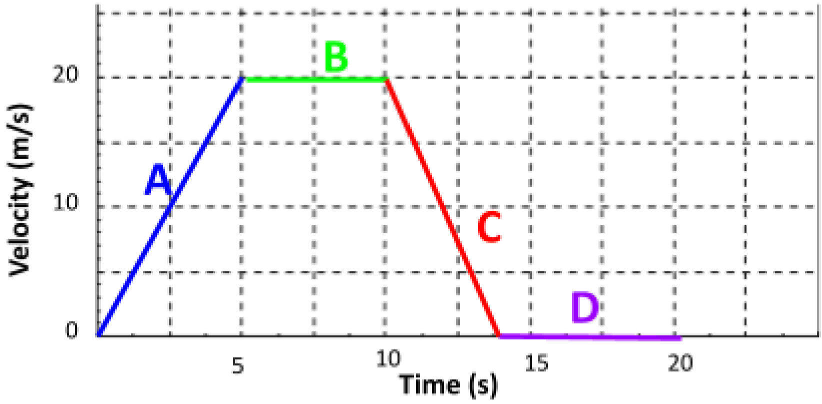
The ambulance is equipped with a GPS tracker that can record its position and time and then present data on a mobile device. Below is the distance-time graph for a portion of its journey.

What comes to mind about the information displayed on the graph?



The ambulance’s GPS tracker can also record its velocity. Below is the velocity-time graph for the same portion of the journey as the previous distance-time graph.
How does one graph relate to the other?

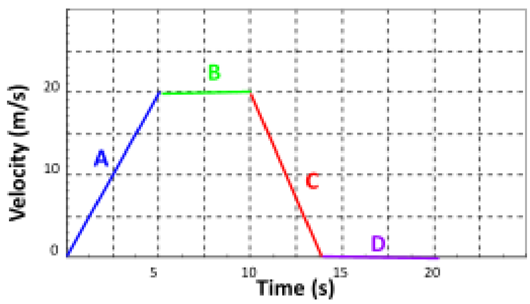




Understanding the Ambulance’s Motion

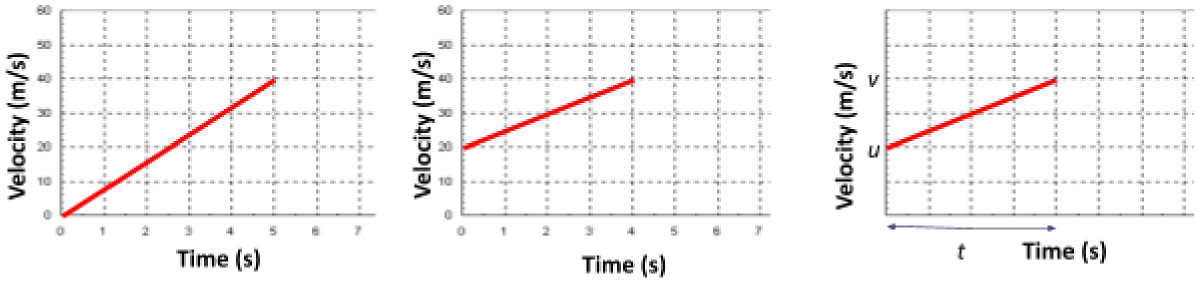
By interpreting the different sections of the ambulance’s velocity-time graph we can learn a lot about it’s motion.

Section of the Graph	Slope	Velocity	Acceleration
A	Positive	Increasing	Positive
B	Zero	Constant	Zero
C	Negative	Decreasing	Negative
D	Zero	Stationary	Zero



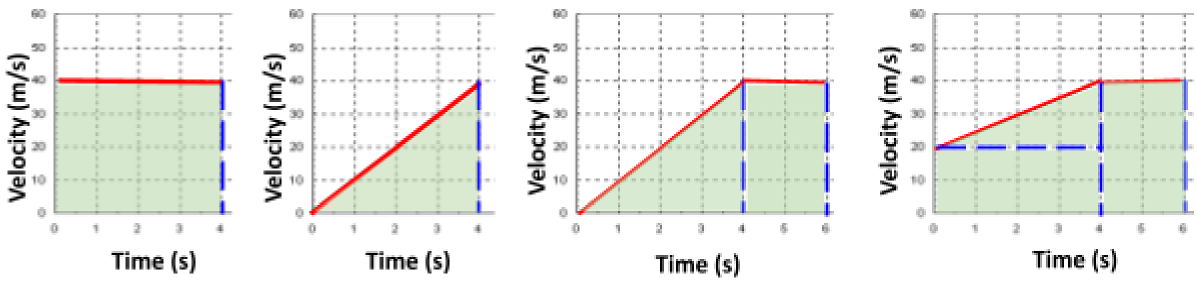
Using the Ambulance’s Motion to Create an Expression for Final Velocity, *v*

Other graphs were produced from the ambulance’s GPS tracker below. Can you create a general expression for final velocity, v, involving the parameters in the graph?



Connecting distance travelled with Velocity-Time graphs

In the velocity-time graphs shown, what conclusions could you make about the distance travelled in each?

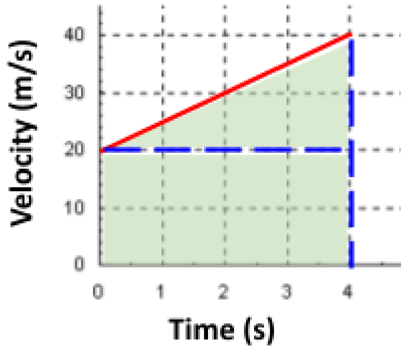


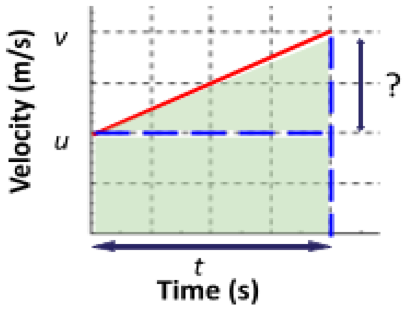
Question: Does your conclusion remain true for an object accelerating from rest @ 10 m/s^2 for 4 sec?

Extension question: Is it true for an object accelerating from 20 m/s @ 5 m/s^2 for 4 sec?

Construction of a General Formula for Distance

Use what you learned in the previous slide to determine the area under the velocity-time graph below.





Area under graph from $t = 0\ s$ to $t = 4\ s$
= Area of + Area of
= (20)(4) + ½ (4)(40 - 20)
= 80 + 40 = 120 m
Could this be generalized into a formula for distance/displacement?

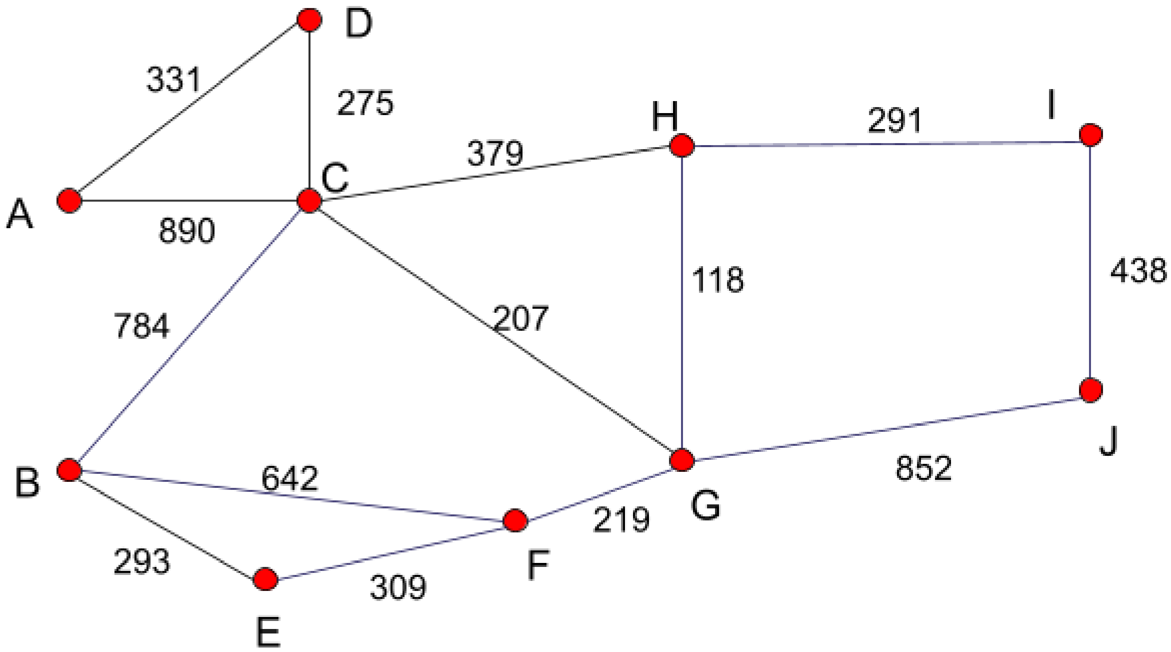
Reflection on Teaching and Learning

What did you notice about the teaching and learning approach used here?

What are some of the benefits for students of using an approach like this?

Authentic Problem - Modelling Response Times

The network below represents the distances, in metres, on a road network. An accident has occurred at J and emergency services are called to the scene. An ambulance starts from A and a Garda motorbike starts from B at the same time with both vehicles trying to reach J as quickly as possible.
What is your understanding of the expression 'real world authentic problem'?
Which vehicle reaches J first?



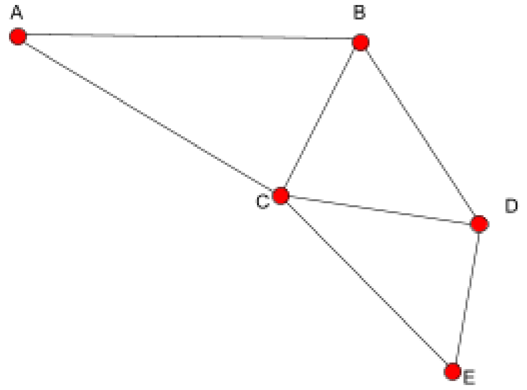
Thinking towards September 2021

How might you integrate mathematical modelling into your existing practice?

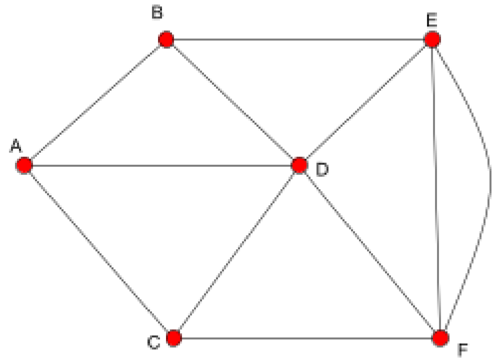
How could you get your students to engage meaningfully with mathematical modelling?

Additional Questions:

Question 1
Use an adjacency matrix to represent this graph



Question 2
Use an adjacency matrix to represent this graph



Question 3
Draw a graph corresponding to the following adjacency matrix

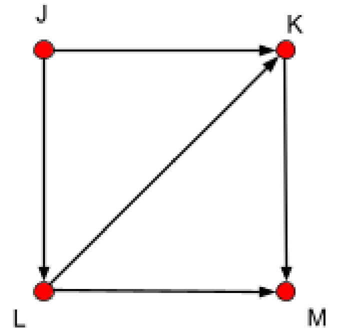
	A	B	C	D	E
A	0	1	0	1	0
B	1	0	1	1	1
C	0	1	0	2	0
D	1	1	2	0	1
E	0	1	0	1	0

Question 4
Draw a graph corresponding to the following adjacency matrix

	A	B	C	D
A	0	1	0	1
B	1	0	2	0
C	0	2	0	1
D	1	0	1	0

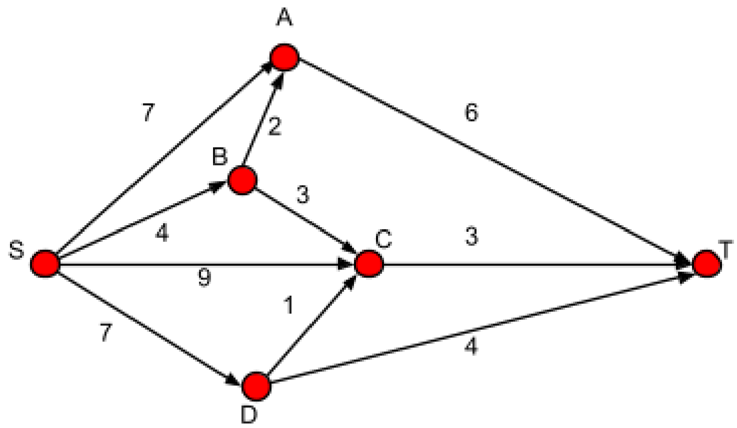
Question 5

In the directed graph below, find the number of ways to get from J to M in exactly 2 moves using adjacency matrix multiplication.



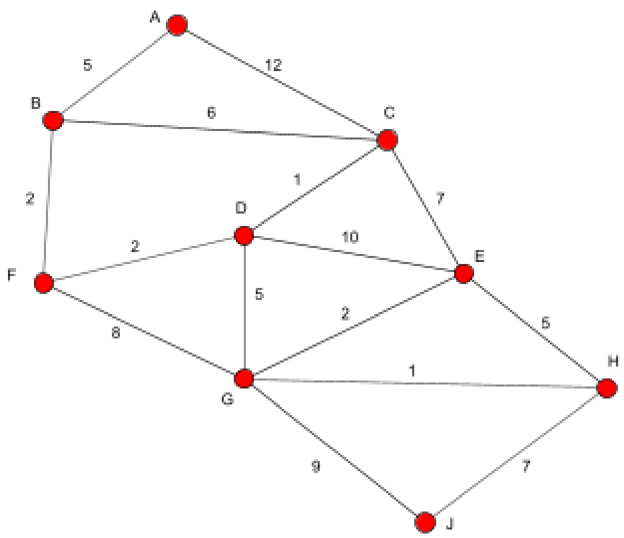
Question 6

Use Dijkstra's Algorithm to find the shortest route from S to T. State the length of your route.



Question 7

Find the shortest path from A to J and state the length of your route.



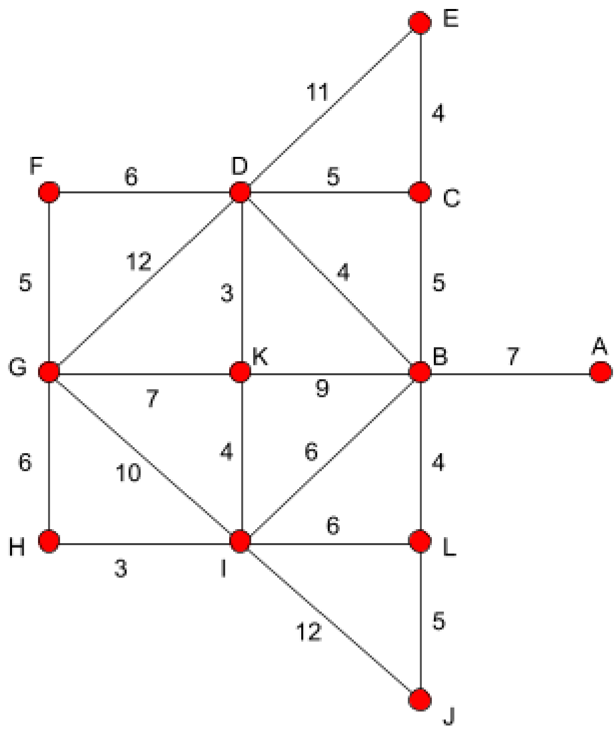
Question 8

The network below shows the times, in minutes, taken by fire trucks to drive along roads connecting 12 places, A, B, …., L.

On a particular day, there are three fire trucks in the area at A, E and J. There is an emergency at G and all three fire trucks drive to G.

(a) (i) Use Dijkstra's algorithm on the network, starting from G, find the minimum driving time for each of the fire trucks to arrive at G.

(ii) For each of the fire trucks, write down the route corresponding to the minimum driving time in your answer to part (a)(i).



Additional Questions: Solutions

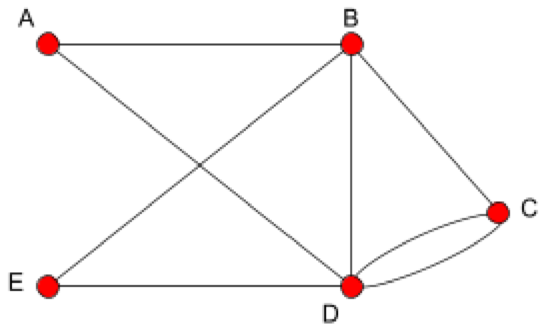
Question 1

	A	B	C	D	E
A	0	1	1	0	0
B	1	0	1	1	0
C	1	1	0	1	1
D	0	1	1	0	1
E	0	0	1	1	0

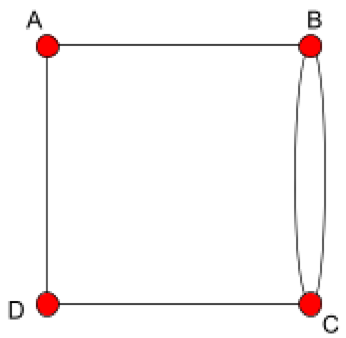
Question 2

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	0	1	1	0
C	1	0	0	1	0	1
D	1	1	1	0	1	1
E	0	1	0	1	0	2
F	0	0	1	1	2	0

Question 3



Question 4

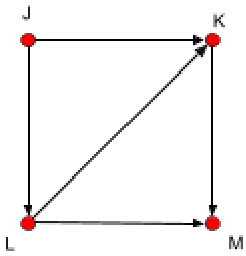


Question 5
The Adjacency matrix is:

	J	K	L	M
J	0	1	1	0
K	0	0	0	1
L	0	1	0	1
M	0	0	0	0

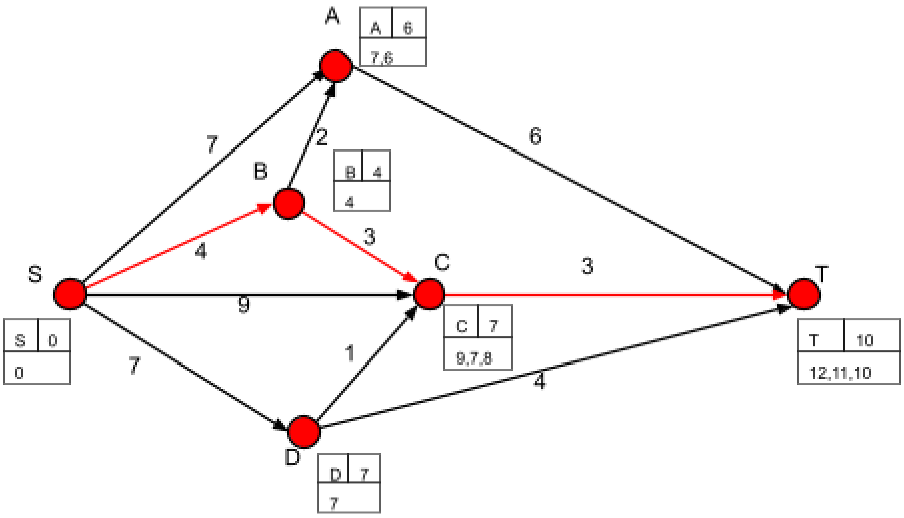
Since we are looking for the number of ways to get from J to M in exactly 2 moves then we must find the (Adjacency Matrix)^2 .

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



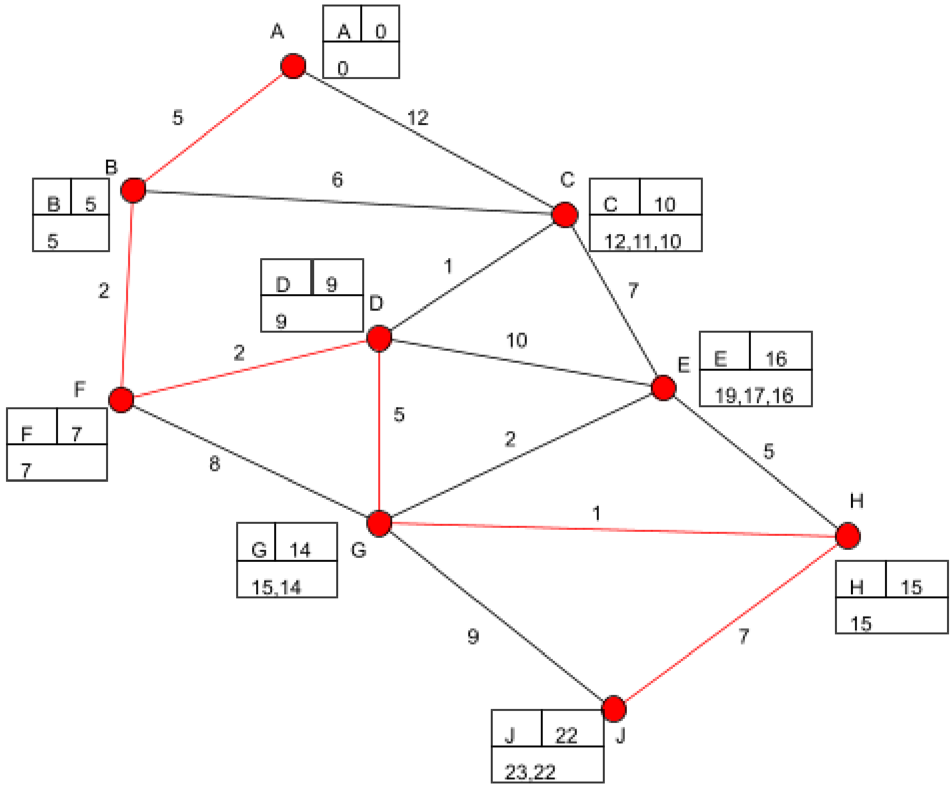
We can see from the resultant matrix above that there are 2 ways to get from J to M in exactly 2 moves.

Question 6



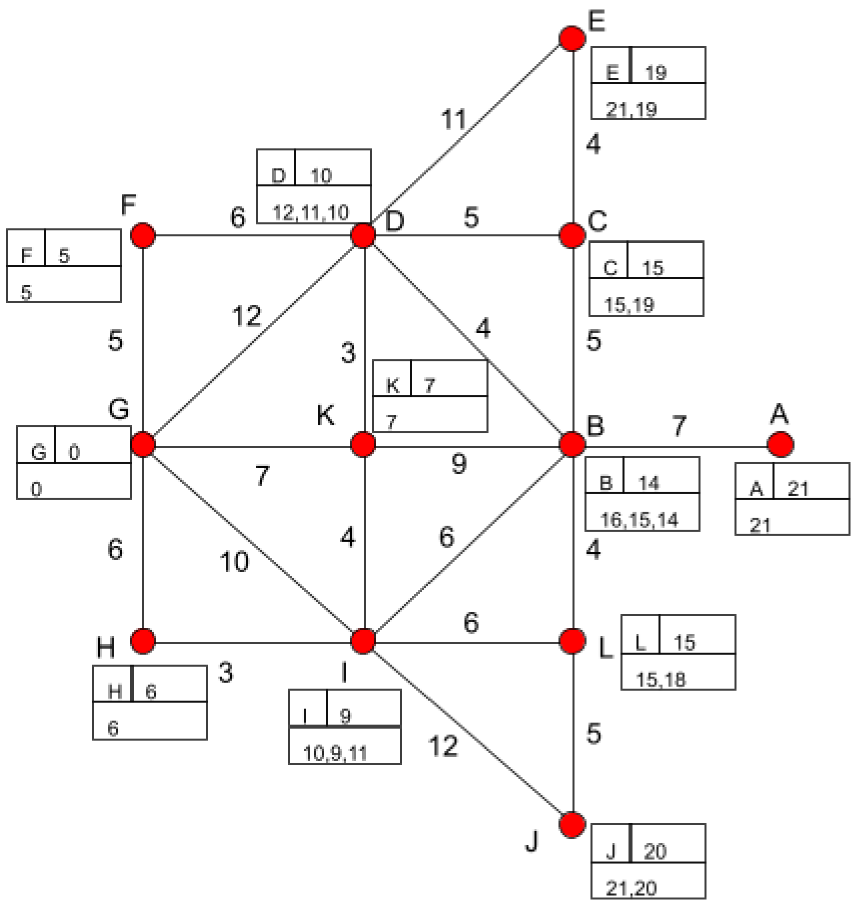
Shortest Route: S-B-C-T Length: 10

Question 7



Shortest Route: A-B-F-D-G-H-J   Length: 22

Question 8



In this case we have 3 fire trucks going to the same destination (G) so since this is an undirected graph treat G as the start and find the shortest route to each destination.

(i) Time taken for fire truck at A = 21 minutes
Time taken for fire truck at E = 19 minutes
Time taken for fire truck at J = 20 minutes

(ii) Shortest routes for fire truck at A: A-B-D-K-G
Shortest routes for fire truck at E: E-C-D-K-G
Shortest routes for fire truck at J: J-L-I-H-G

Learning Log:

Appendix:

Glance Card