The Elevator Problem is from GAIMME by the Society for Industrial and Applied Mathematics (SIAM).

THE ELEVATOR PROBLEM

The *Elevator Problem* can be used with many different classes. Naturally, the more advanced the class, the more that can be expected in the solution. The problem has no mathematical prerequisites and has been used successfully with Algebra 1, Precalculus, and post-Calculus students. After spending several class periods on the problem, one Precalculus student commented, "Never before, in all my math experiences, had I seen a problem as open ended and varying as this one. Working on a problem like this with no obvious answer and many different options was a wholly new experience for me. This problem helped me visualize the role math could and most likely will play in my future."

If used as a motivation or application of standard content, an algebra class could focus on developing a simple linear function of two variables, while a class that has studied some probability might focus on the problem as an application of combinatorial probabilities. It can also serve as a vehicle for discussing simulation models involving probability. The *Elevator Problem* serves equally well as a totally open problem unrelated to the current content being studied. Problems like this one demonstrate that modeling at the high school level often involves a sophisticated application of very elementary concepts.

The problem focuses attention to the importance of making reasonable, simplifying assumptions. Like many modeling problems, the *Elevator Problem* can be made as simple or complex as the teacher desires by altering a few of the parameters in the problem and by the amount of scaffolding and student–teacher conversation in the presentation of the problem. Each teacher can determine how difficult the problem should be and how far into the problem they want their students to go, based on their goals for the activity.

> ***Problem Statement*** Walton and Davidson in the Spode Group's wonderful volume Solving Real Problems with Mathematics, Volume 2 present the elevator problem in the form of a series of memos between your boss and you (the student), and between you and your assistant discussing the problem of late arrivals at work.

MEMO #1

From: Your Boss

To: You

Re: Late Arrivals

I have received numerous complaints that large numbers of our employees are reaching their offices well after 9:00 a.m. due to the inability of the present three elevators to cope with the rush at the start of the day. In the present financial situation it is impossible to consider installing any extra elevators or increasing the capacity of existing ones above the current ten persons. Please investigate and let me have some possible solutions to the problem with an indication of their various advantages and disadvantages.

MEMO #2

From: You

To: Your Assistant

Re: Late Arrivals

Can you find out:

1. How long the elevators take to get between floors and how long they stop for?
2. How many people from each floor use the elevator in the morning?
3. How many people were late this morning?

MEMO #3

From: Your Assistant

To: You

Re: Answers to your questions

1. The elevators appear to take 5 seconds between each floor, an extra 15 seconds for each stop, and another 5 seconds if the doors have to reopen. It also seems to take about 25 seconds for the elevator to fill on the ground floor.
2. The number of workers on each floor are:

FLOOR	G	1st	2nd	3rd	4th	5th
NUMBER	0	60	60	60	60	60

3. About 60 people were late today.

MEMO #4

From: You

To: Your Boss

Re: Solution to the problem with advantages and disadvantages

?

In the problem posed, we see that the building has 5 floors (1-5) that are occupied. The ground floor (0) is not used for business purposes. Each floor has 60 people working on it and there are 3 elevators (A, B, and C) available to take these employees to their offices in the morning. Each elevator holds 10 people and takes approximately 25 seconds to fill on the ground floor. The elevators then take 5 seconds to travel between floors and 15 seconds on each floor on which it stops.

Have students discuss the situation and think about what might be creating the late arrivals and how the elevator system might be changed to address the problem. As one aid, you might have them discuss the elevator signs from a hotel (Figure C.1).

Could such a system be utilized during the morning rush (8:30 am - 9:00 am)?

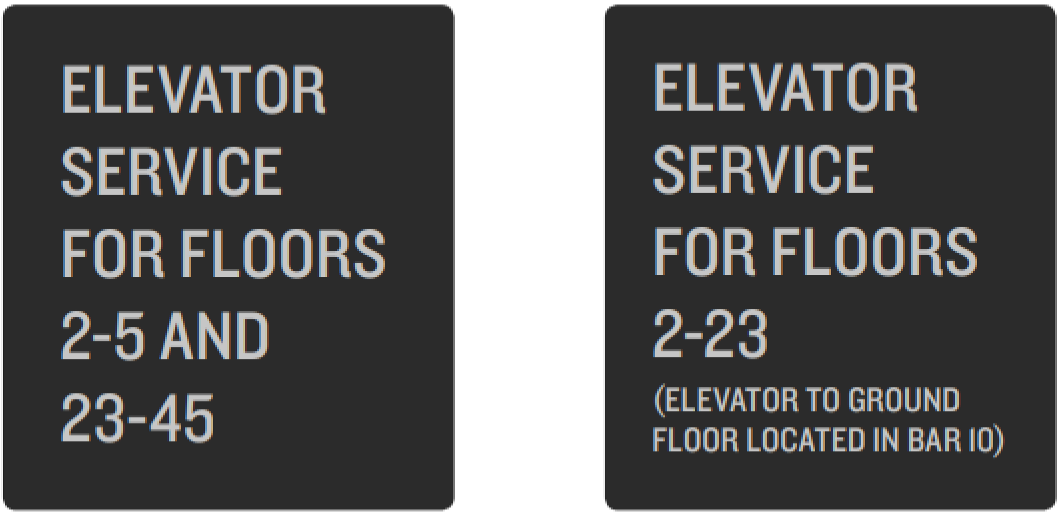

FIGURE C.1: HOTEL ELEVATOR SIGNS AS AN EXAMPLE ELEVATOR SYSTEM

MAKING REASONABLE SIMPLIFYING ASSUMPTIONS

Before proceeding to a discussion of the problem, it will be helpful to look at some basic assumptions that can be made to simplify and clarify the problem to be solved.
Students must make some assumptions about the current process to set up their work. The assumptions need to be couched in the reality of the problem setting, but allow for some simplicity in modeling the process. The assumptions must also be consistent with, or at least not contradict the information at hand.

Students will often try to solve the problem by requiring employees to arrive at specific times (those working on the fourth floor should arrive at 8:47, for example), or requiring those working on the lower floors to take the stairs, but to the extent possible, we would like to alter the elevators, not the habits of the individuals. Encourage the modelers to make no changes to the worker's current morning routine. Changing the daily patterns of the employees will create problems between the workers and management that should be avoided. This charge means we need to make some assumptions about what that current routine could likely be.

For example, we could not assume all employees arrive at 5 minutes to 9:00, since, if that were the case, we would have many more than 60 employees late. Since the boss did not write to complain about employees arriving at the elevators too late to get to work on time, we can assume that employees arrive at a time that should allow them to get to work, but operation of the elevators keeps them from it.

Note: Have students discuss plausible reasons for workers arriving late for work because of the slowness of taking the elevators.

Assumption 1: We assume that at some time prior to 9:00, employees begin arriving, and, once the arrival process starts, there is a steady stream of employees waiting to take the elevators. The memo does not mention workers arriving at the elevators late, but arriving in their offices late. This assumption gives us a realistic problem that might have a solution involving the elevators.

Assumption 2: We assume that the current situation is that each elevator carries employees to all floors, necessitating stops on each floor on each trip. When in a hurry to get to their office, we expect that the workers will take the first available elevator. This creates a mixed group filling each elevator, which may be the reason the trips take too long.

Assumption 3: Since the issue is getting employees to their floor efficiently, we assume that the only use of the elevators between 8:30 and 9:00 is in getting from the ground floor to the appropriate floor for their job. We will ignore the possibility that workers will be moving between floors (going from 4th to 2nd, for example) during this time.

While this assumption is clearly unrealistic, we have no information about how prevalent this movement is and consequently no good way to incorporate it in the model. Moreover, trying to do so initially makes the problem much too complicated. Once students have created an initial model, they may be able to add this piece of the problem to it, but it is very important that students keep their first models as clean and simple as possible. The modeling cycle allows for successive refinement to deal with smaller, but still important components that have been initially ignored. Our first model will focus only on moving employees from the garage up to their floor.

Assumption 4: Elevator doors do not re-open. Since everyone is anxious to get to work on time, they exit the elevators efficiently. In a first model, considering small issues like this can make the problem more difficult and can hide some important features.

A SIMPLE MODEL OF ELEVATOR MOVEMENT

One way to approach this problem is to consider the worst case scenario. The worst that can happen is for each elevator to always have at least one person from each floor on it. This means the elevator will have to make the longest possible trip each time. Since there are 60 people on 5 floors, there are 300 employees to take to their offices. On average, we would expect that each of the elevators will carry 100 people. Since the capacity of each elevator is ten people, each elevator will make 10 trips. How long will each of these trips take? The Figure C.2 below illustrates one complete trip. Since no one is being picked up or let off, the elevators do not stop on the way down.

The total time of the trip is given by $T = 25 + 5\,(10) + 5\,(15) = 150$ seconds per trip.

Since each elevator makes 10 trips and there are three moving simultaneously, the total time is 1500 seconds, or about 25 minutes. Some students may note that we don't need to count the final trip back to the ground floor, and use 1475 seconds as their estimate.

In the calculation above, we have made an important assumption. If we have a crowd of workers waiting for the elevators, then when the elevator door opens on the ground floor, there are 10 employees ready to get on. Each elevator is full at the start of its ride.

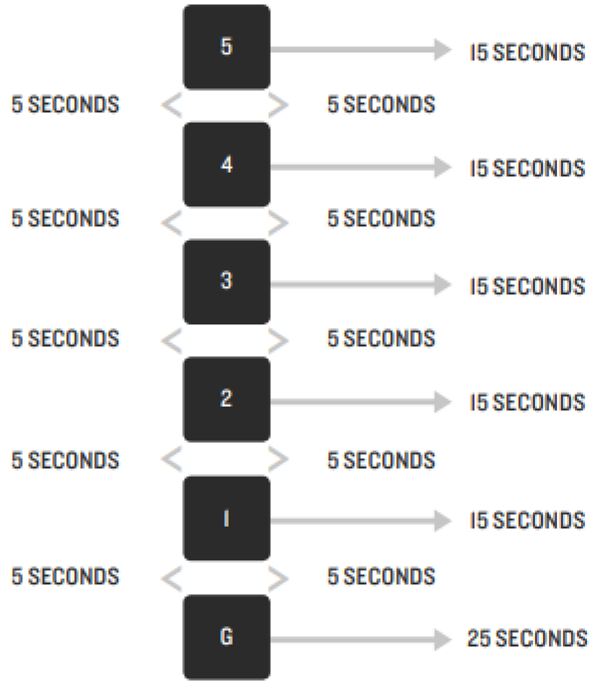

FIGURE C.2: GRAPHICAL MODEL OF AN ELEVATOR TRANSIT.

Assumption 5: Every elevator is full with 10 workers getting on at the ground floor.

Note: Once we have a solution to the nice problem using 10 people per trip, we can recalculate everything using an average of 8 or 9 to see how our model performs. But we want to keep the first model neat and clean to see what the important issues are.

MODELING THE CURRENT STATE

We know that 60 workers were late this morning. Sixty workers represents six elevator trips under our assumptions. With three elevators moving simultaneously, each elevator must make two trips after 9:00. The total time to move all 60 workers to their floor under our initial assumptions is (2 trips)(150 sec/trip) = 300 seconds, or 5 minutes. We need to cut around 300 seconds from the 1500 seconds our simple models predicts it takes to move everyone upstairs. Our target is a maximum of 1200 seconds. Is this possible?

Since it takes approximately 1500 seconds for everyone to be taken to their offices, and your assistant found that 240 (80%) of the employees arrived on time, we can conclude that people begin arriving at a time that allows only 80% of the required 1500 seconds to occur before nine o'clock. That is, they begin to arrive approximately 20 minutes before 9:00.

Rerouting the Elevators It was suggested earlier, in the discussion about the hotel elevators, that rerouting some of the elevators during the rush might be helpful to the company. Let's try one example: what would happen if two elevators went to floors 1, 2, and 3, and one elevator went to floors 4 and 5? How long would this configuration take to get everyone to the proper floor?

We could use the diagram above to model the situation for each elevator, or we can create an algebraic expression whose value gives the time of the trip more simply.

Have the students discuss what the total transit time of an elevator depends upon. What determines how long it takes? Students will recognize that the transit time depends upon two things, the number (N) of floors at which the elevator stops, and the highest floor (F) to which the elevator travels. The transit time is
$T = 25 + 10F + 15N$ seconds per trip.

In the example given above, the trip to the first, second and third floors takes
$T = 25 + 10\ (3) + 15\ (3) = 100$ seconds per trip,

while the trip to the fourth and fifth floors takes
$T = 25 + 10\ (5) + 15\ (2) = 105$ seconds per trip.

As shown in Table C.1, with this configuration, the slowest elevator takes 1260 seconds, a reduction in total transit time of 4 minutes over having all elevators travel to all floors. Can we do better? Can we reduce the expected time by our goal of six minutes?

As shown below, if we let one elevator travel to floors 1 and 2 and the other two travel to floors 3, 4 and 5, then we can reduce the total transit time to 1080 seconds, or 7 minutes less than having the elevators travel to all floors. That meets our goal, but are there even better arrangements?

The problem can be stopped here if desired. The focus of the problem may be on setting the simplifying assumptions, generating the time equation $T = 25 + 10F + 15N$, and using it in an interesting, problem solving investigation. This problem lends itself well to group work, as students working in groups can quickly investigate other promising combinations of elevators. We based our calculations on the worst case assumption that every elevator stops at every floor to which it can travel and the best case assumption that every elevator would be full. We have a solution that does not require the employees to alter their preferred arrival time, only to take a specific elevator between 8:30 and 9:00 in the morning. After 9:00, the elevators are available for use by everyone.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A & B	1, 2, 3	180	9 each	100	900
C	4, 5	120	12	105	1260

TABLE C.1: TOTAL TRANSIT TIME WITH ELEVATOR C TO FLOORS 4 AND 5.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A	1, 2	120	12	75	900
B & C	3, 4, 5	180	9 each	120	1080

TABLE C.2: TOTAL TRANSIT TIME WITH ELEVATOR A TO FLOORS 1 AND 2.

Now that students see how the problem works, they can include such details as having an average of 9 workers per trip, consideration of trips that, just by chance, don't stop on every floor, or other more realistic components. We will consider some variations of our assumptions later.

PRESENTING THE PROBLEM

The problem as described so far, has been designed for students new to modeling. This problem serves well as a first modeling adventure, so a whole-class conversation setting the stage and supporting the development of assumptions is helpful in getting the students on a reasonable track.

As mentioned in *Driving for Gas*, asking specific questions can help the student focus on important issues. The goal of this questioning is not to tell students what to do, but to raise questions that, as students think about and discuss them, illuminate the important issues for the students. It is a form of scaffolding for students. For example, the list of questions below has been used in an all-class discussion with Precalculus students who were engaging this simple version of the *Elevator Problem* as their first mathematical modeling experience. The goal of the questioning is to have students think about and come up with the important considerations to be addressed in the model. It is important that the questions not be overly directive.

- In some buildings, all of the elevators can travel to all of the floors, while in others, the elevators are restricted to stopping only on certain floors. Why?
- What is the advantage of having elevators which travel only between certain floors?
- Suppose it is a holiday and only 5 people come to work today. Each person works on a different floor, and they all ride the same elevator. How long will it take for everyone to get to work?
- Now suppose that 5 people come to work and these five people do not all work on different floors. How long will it take for everyone to get to work?
- Why was the last question harder to answer than the prior question? What assumptions will you need to make in order to simplify the problem?

Such questions can convince students to make some reasonable simplifying assumptions about how the elevators fill on the ground floor. Typical student assumptions include:

- There will be 10 people waiting for the elevator on the ground floor and the elevator will fill to capacity (10 people) for each trip.
- If an elevator can go to a floor, then it will go to that floor on each trip.
- No one takes the stairs.
- No one uses the elevator to go down during this time (or if they do, it does not impact the time for the elevator to complete its trip).
- The elevator doors don't have to re-open on any floors.

These are the assumptions used in the basic solution presented above. The first two assumptions will seem unrealistic to students, but the goal of assumption is to simplify the problem to a tractable form given the abilities of the students. Both of these assumptions can be modified once the initial simple problem has been solved. This is a difficult issue for some students. As one student said in his evaluation of the problem, "It is difficult for us to let go of the details". Students may initially feel like making assumptions is cheating somehow. Once through a modeling cycle and they see that they can return to the assumptions and modify them to make them more realistic, they accept that good assumptions can be unrealistic in their first instance.

After agreement is reached on the basic assumptions to be used, some new thought-provoking questions might be asked.

- If all elevators go to all floors, how long will it take everyone to get to work?
- If 80 people were late using the unrestricted elevators, approximately what time did the employees begin arriving at the ground floor?

Once this last question is answered, the students have a target for their task. Beginning at the same arrival times, students need to reduce the total time to get everyone to the appropriate floor by however long it takes for the 80 late employees to reach their destination.

- Reassign the elevators to transport the employees to their offices as quickly as possible. What arrangement produces the shortest time? If this arrangement had been used today, would everyone have arrived at their floor on time?

Once completed, the students can be led to reflect on their work and comment on the strategies they use that could be applied to future mathematical modeling adventures. Examples include:

- We used a simple case to understand the structure of the problem.
- We drew a diagram to help us visualize the scenario.
- We thought about what made the problem hard to help us figure out simplifying assumptions.
- We considered the *worst case scenario* (if an elevator can go to a floor, it will go to that floor) and solved this rather than trying to think about all of the different possibilities.
- We found a solution that worked, then modified it to see if we could improve it.
- We had to make sure our solution was realistic. (Sometimes "mathematically optimal" is not optimal in the real world.)

This last comment concerning mathematically optimal solutions is important. Students can easily see that having all workers on a floor arrive at a specific time and all ride the same elevator, so it only makes one stop, would be much more efficient. They can create a much "better" result than the one described above. But, that solution would not work well in the real world.

As mentioned earlier, each teacher can determine how difficult the problem should be and how far into the problem they want their students to go, based on their goals for the activity. One attribute of good modeling problems is that they can be extended in many directions if students have the necessary mathematics in their background. Variations of this problem have been used successfully with students from Algebra I to post-Calculus students in AP Statistics.

SECOND ITERATION AND IMPROVING THE MODEL

The full elevator assumption may be true at the end of the process when everyone is eager to get to work, but it may not be true early in the process. Suppose, on average, there are only 9 workers riding each elevator. In this case we have a couple of extra trips that are required. This is an easy change to make, now that we see how the process works.

The total time of the trip is still given by
$T = 25 + 5\,(10) + 5\,(15) = 150$ seconds per trip. seconds per trip.

Now there will be two elevators making 11 trips and one making 12 with only a few workers on the last trip (see Figures C.3 and C.4). Students may decide to ignore the final trip. If we can reduce the number of late workers from 60 to 4 or 5, we have essentially solved the problem. Management will likely not be too concerned about several workers arriving a minute after 9:00. Using the formula found earlier, students find the 11 trips take 1650 seconds (27.5 minutes) and the last elevator takes 30 minutes. Since 60 workers were late, this is three trips or 450 seconds. Our new target is around 1200 seconds.

The solution with one elevator going to floors 1 and 2 and the others to 3, 4, and 5 continues to work in this setting. If we reduce the average number to 8 per trip, then we have 12 or 13 trips per elevator (see Figure C.5). The time for these trips is between 30 and 33 minutes. Our new target is around 24 minutes or 1440 seconds.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A & B	1, 2, 3	180	10 each	100	1000
C	4, 5	120	14	105	1470

TABLE C.3: TOTAL TRANSIT TIME WITH ELEVATOR C TO FLOORS 4 AND 5.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A	1, 2	120	12	75	900
B & C	3, 4, 5	180	9 each	120	1080

TABLE C.4: TOTAL TRANSIT TIME WITH ELEVATOR A TO FLOORS 1 AND 2.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A	1, 2	120	15	75	1125
B & C	3, 4, 5	180	12 each	120	1440

TABLE C.5: TOTAL TRANSIT TIME WITH ELEVATOR A TO FLOORS 1 AND 2.

We just barely make it. If the average was reduced to 7 per trip, which seems unrealistically low, then our solution would not work (but perhaps another variation would). By recomputing with nine, then eight, passengers per trip, we are testing the sensitivity of our model to the assumption that each elevator is full. In this case, as long as there are at least 8 passengers, on average, per trip, our solution will work. We say that our model is fairly insensitive to a change in this assumption.

THIRD ITERATION AND PROBABILITY

How realistic is our worst case assumption that an elevator will go to every floor possible? For example, how likely is it that an elevator with 10 passengers carries none of the 60 employees working on the fifth floor?

Since all of the floors have the same number of workers and thus, are equally likely on an elevator, students can easily simulate this probability with their calculators. If each student uses the random integer command to create 10 random integers from 1 to 5 (randInt(1,5,10) on the TI-84), they can simulate the floors that 10 workers in an elevator will go to. If they look through the ten integers to count how often none of them are a 5, they should see, by repeating the process 20 times and comparing results, that about 1 in 10 trips has no one going all the way to the 5th floor. So, instead of taking 150 seconds, this trip would take only 125 seconds. Since there are equal numbers of employees on each floor, the results for those floors are similar.

If students have studied probability, they could do a binomial approximation consistent with the random integer simulation by computing the probability that all 10 workers in an elevator come from floors 1-4 by
$(4/5)^{10} \approx 0.107$.

If they have studied combinations, they could compute the actual probability as

$$\frac{\binom{240}{10}\binom{60}{0}}{\binom{300}{10}} \approx 0.103$$

Using simulations or calculations, students can expect that of the ten trips an elevator makes, one of them would go only to the first 4 floors and take only 125 seconds for the trip. In a similar manner, we find that 10% of the transits don't stop on floor 1, 10% don't stop on floor 2, 10% don't stop on floor 3, and 10% don't stop on floor 4. Each of these transits takes 135 seconds. This analysis is also not quite right, since it only applies to the first elevator trip. Once the first trip is made, the number of workers on each floor changes. But this gives a reasonable approximation that can be useful.

More advanced students might also consider if some of the 10% that don't stop on the 5th floor also don't stop on the 4th. Simulations will show that in 10 trips, this happens so rarely that it need not be considered. The probability that an elevator with 10 people has no one from either the 4th or 5th floors is

$$\frac{\binom{240}{10}\binom{60}{0}}{\binom{300}{10}} \approx 0.005$$

This means that of the 10 trips, we would expect none of them to miss two floors.

Indeed, all two floor combinations have this same probability, since all floors have the same number of people. Three floor combinations are even less likely. In 10 trips, it is reasonable to think that some of the elevator transits will miss one floor, but none will miss more than one. Of the 10 trips, we expect that five go to all floors, while the remaining five each miss one floor. So a more realistic expected total travel time is 5 (150) + 1 (125) + 4 (135) = 1415 seconds, MEMO #3

From: Your Assistant

To: You

Re: Answers to your questions

1. The elevators appear to take 4 seconds between each floor, an extra 10 seconds for each stop, and another 5 seconds if the doors have to reopen. It also seems to take about 15 seconds for the elevator to fill on the ground floor.
2. The number of workers on each floor are:

FLOOR	G	1st	2nd	3rd	4th	5th	6th
NUMBER	0	80	80	40	80	20	20

3. About 70 people were late today.

In this scenario, there will be 32 elevator trips if all elevators are full. Clearly, with only 20 employees on the 5th and 6th floors, not all of them will stop at each floor. In this setting, the probability computations are essential. Also, with different number of workers on the floors and with the addition of a 6th floor, there are more combinations of elevator trips for students to consider. Another variation is to add a fourth elevator to again increase the number of options students should consider.