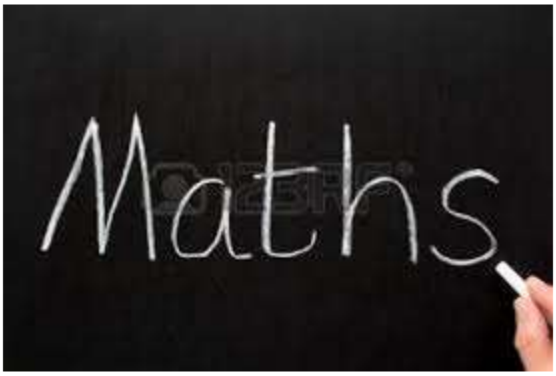
PDST

Professional Development Service for Teachers | An tSeirbhís um Fhorbairt Ghairmiúil do Mhúinteoirí

Practical Approaches to Developing **Mental Maths Strategies** for Addition and Subtraction



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This manual has been designed by members of the Professional Development Service for Teachers. Its sole purpose is to enhance teaching and learning in Irish primary schools and will be mediated to practising teachers in the professional development setting. Thereafter it will be available as a free downloadable resource on www.pdst.ie for use in the classroom. This resource is strictly the intellectual property of PDST and it is not intended that it be made commercially available through publishers. All ideas, suggestions and activities remain the intellectual property of the authors (all ideas and activities that were sourced elsewhere and are not those of the authors are acknowledged throughout the manual).

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Introduction

This booklet is intended to support teachers in developing addition and subtraction mental maths strategies in their classrooms. It has been designed to accompany the PDST Mental Maths workshops.

The booklet explores the key properties of number and number relationships relating to addition and subtraction, outlining practical approaches to developing an understanding of these. It also explores background knowledge for teachers and fundamental facts in relation to mental maths.

A range of concrete, practical activities that will support pupils in their development of addition and subtraction mental maths strategies is also outlined.

Finally, a selection of engaging and enjoyable activities to consolidate learning and provide opportunities for pupils to master addition and subtraction facts is included.

Background Knowledge for Teachers

Stages of Progression

Arthur Baroody (2006, p.22) identifies three stages through which children progress in order to acquire the basic facts of addition, subtraction, multiplication and division:

1. Counting Strategies: using object counting (for example blocks or fingers) or verbal counting to determine the answer. For example, with 4 + 7 pupil starts with 7 and counts on verbally 8, 9, 10, and 11.

2. Reasoning Strategies: using known information to logically determine an unknown combination. For example, with 4 + 7 pupil knows that 7 + 3 is 10, so 7 + 4 is one more, 11.

3. Mastery: efficient (fast and accurate) production of answers. For example, with 4 + 7 pupil quickly responds, ‘It’s 11; I just know it.’

Uses of Mental Calculation

The following six aspects of mathematics require the use of mental calculation (Crown 2010).

Recalling Facts:	**Applying Facts:**
• What is 3 add 7? • What is 6 × 9? • How many days are there in a week?... in four weeks? • What fraction is equivalent to 0.25? • How many minutes in an hour? ... in six hours?	• Tell me two numbers that have a difference of 12. • If 3 × 8 is 24, what is 6 × 0.8? • What is 20% of €30? • What are the factors of 42? • What is the remainder when 31 is divided by 4?

Hypothesising or Predicting:	**Designing and Comparing Procedures:**
• The number 6 is 1 + 2 + 3, the number 13 is 6 + 7. Which numbers to 20 are the sum of consecutive numbers? • Roughly, what is 51 times 47? • On a 1 to 9 keypad, does each row, column and diagonal sum to a number that is a multiple of 3?	• How might we count a pile of sticks? • How could you subtract 37 from 82? • How could we test a number to see if is divisible by 6? • How could we find 20% of a quantity? • Are these all equivalent calculations: 34 – 19; 24 – 9; 45 – 30; 33 – 20; 30 – 15?

Interpreting Results:	**Applying Reasoning:**
• So what does that tell us about numbers that end in 5 or 0? • Double 15 and double again; now divide your answer by 4. What do you notice? Will this always work? • I know 5% of a length is 2 cm. What other percentages can we work out quickly?	• The seven coins in my purse total 23c. What could they be? • In how many different ways can four children sit at a round table? • Why is the sum of two odd numbers always even?

Fundamental Facts for Addition and Subtraction

The Commutative Property

Numbers can be added in any order for example 7 + 12 = 12 + 7. This property is quite useful in problem solving, mastering basic facts and in mental maths. Therefore, it is important to spend some time helping pupils to construct the relationship (Van deWalle, p.153). This relationship applies to addition but not to subtraction. In subtraction, order does matter for example, $5 - 3 \neq 3 - 5$. But a series of consecutive subtractions can be taken from a number in any order, for example, 15 – 3 – 5 = 15 – 5 – 3.

The Associative Property

When three or more numbers are added together, they can be added in any order. It is a useful property for children to understand as it allows them to add combinations of numbers they know. In practice, two of the numbers have to be added together or associated first, and then a third number is added to the associated pair for example 7 + 5 + 3 = (7 + 5) + 3 = 7 + (5 + 3) = (7 + 3) + 5. (Crown 2010)

Inverse Relationship

Every addition calculation can be replaced by an equivalent subtraction calculation and vice versa for example 5+7=12 implies 5=12-7 and 7=12-5. In the same way, 13 – 6 = 7 implies 13 = 7 + 6 and 6 = 13 – 7. (Crown 2010)

Pupils should encounter subtraction in the following forms, developing an understanding of each:

Subtraction as Deducting

If the quantity of a set is reduced, how many will be left? For example, *I had 10 sweets. I ate 3. How many sweets are left?* (Maths Curriculum 1999)

Subtraction as Complementing

If the set is not yet full, how many more will be needed to fill it? For example, *There are 10 stickers in a set. I have 4. How many more do I need to make a full set?* (Maths Curriculum 1999)

Subtraction as Difference

When comparing sets, the pupil identifies how many more/how many less are in each set. For example, *I have 12 crayons. Mary has 6 crayons. How many more crayons have I? How many less crayons has Mary?* (Maths Curriculum 1999)

Possible Pupil Misconceptions

The Zero Property for addition states that when adding zero to a number, the number does not change. It is common for pupils to feel that 4 + 0 must make more than 4 because usually, when we add, we get a bigger number as our answer.

Teaching and Learning

Suggested Addition and Subtraction Strategies

Counting forwards and backwards	Doubles/near doubles	Facts of 10
Bridging through ten(s)	Reordering	Partitioning by place value
Compensating	Bridging through 60 (time)	**Subtraction** • Think addition • Keeping a constant difference

Instructional Framework

Table 1.1 on the following page illustrates a framework for advancing mathematical thinking. Although it does not explicitly refer to concrete materials or manipulatives, the use of these are often a prerequisite for developing mathematical thinking and can be used as a stimulus for this type of classroom discourse.

Table 1.1 Strategies for Supporting and Developing Mathematical Thinking

Eliciting	Supporting	Extending
Facilitates pupils' responding Elicits many solution methods for one problem from the entire class *e.g. "Who did it another way?; did anyone do it differently?; did someone do it in a different way to X?; is there another way of doing it?"* Waits for pupils' descriptions of solution methods and encourages elaboration Creates a safe environment for mathematical thinking *e.g. all efforts are valued and errors are used as learning points* Promotes collaborative problem solving *Orchestrates classroom discussions* Uses pupils explanations for lesson's content Identifies ideas and methods that need to be shared publicly *e.g. "John could you share your method with all of us; Mary has an interesting idea which I think would be useful for us to hear."*	*Supports describer's thinking* Reminds pupils of conceptually similar problem situations Directs group help for an individual student through collective group responsibility Assists individual pupils in clarifying their own solution methods *Supports listeners' thinking* Provides teacher-led instant replays *e.g. "Harry suggests that ...; So what you did was ...; So you think that ...".* Demonstrates teacher-selected solution methods without endorsing the adoption of a particular method *e.g. "I have an idea ...; How about ...?; Would it work if we ...?; Could we ...?".* *Supports describer's and listeners' thinking* Records representation of each solution method on the board Asks a different student to explain a peer's method *e.g. revoicing (see footnote on page 8)*	*Maintains high standards and expectations for all pupils* Asks all pupils to attempt to solve difficult problems and to try various solution methods *Encourages mathematical reflection* Facilitates development of mathematical skills as outlined in the PSMC for each class level *e.g. reasoning, hypothesising, justifying, etc.* Promotes use of learning logs by all pupils *e.g. see Appendix A for a sample learning log* *Goes beyond initial solution methods* Pushes individual pupils to try alternative solution methods for one problem situation Encourages pupils to critically analyse and evaluate solution methods *e.g. by asking themselves "are there other ways of solving this?; which is the most efficient way?; which way is easiest to understand and why?".* Encourages pupils to articulate, justify and refine mathematical thinking *Revoicing can also be used here* Uses pupils' responses, questions, and problems as core lesson including student-generated problems *Cultivates love of challenge*

This is adapted from Fraivillig, Murphy and Fuson's (1999) Advancing Pupils' Mathematical Thinking (ACT) framework.

Classroom Culture

Creating and maintaining the correct classroom culture is a pre-requisite for developing and enhancing mathematical thinking. This requires the teacher to:

- cultivate a 'have a go' attitude where all contributions are valued;
- emphasise the importance of the process and experimenting with various methods;
- facilitate collaborative learning through whole-class, pair and group work;
- praise effort;
- encourage pupils to share their ideas and solutions with others;
- recognise that he/she is not the sole validator of knowledge in the mathematics lesson;
- ask probing questions ;
- expect pupils to grapple with deep mathematical content;
- value understanding over 'quick-fix' answers; and
- use revoicing[1] (reformulation of ideas) as a tool for clarifying and extending thinking.

In this type of classroom pupils are expected to:

- share ideas and solutions but also be willing to listen to those of others; and
- take responsibility for their own understanding but also that of others.

[1] [1] Revoicing is 'the reporting, repeating, expanding or reformulating a student's contribution so as to articulate presupposed information, emphasise particular aspects of the explanation, disambiguate terminology, align students with positions in an argument or attribute motivational states to students' (Forman & Larreamendy-Joerns, 1998, p. 106).

Problems and Solutions in Supporting and Developing Mathematical Thinking

Table adapted from Chapin, O’Connor and Anderson (2013) Classroom Discussions In Math: A Teacher's Guide for Using Talk Moves to Support the Common Core and More. Math Solutions

Problem	Solution
Only one pupil volunteers to talk – the pupil who always has the right answer	Try the **Eliciting** moves: *wait time, turn and talk* or *stop and jot,* after *turn and talk* or *stop and jot*, ask ‘*will you share that with the class?’*
Problem	**Solution**
The pupils answer is wrong, and the teacher isn’t clear on what the pupil is thinking	Try the **Eliciting** move *say more….*
Problem	**Solution**
What the pupil is trying to say just gets more confusing as he or she attempts to say more	Try the **Supporting** move revoicing, *so you are saying?*
Problem	**Solution**
The teacher isn’t sure that the other pupils have heard and understood a pupil’s idea	Try the **Supporting** move revoicing, *who can revoice…?*
Problem	**Solution**
The teacher feels that a pupil needs to take his or her reasoning deeper. How can this happen?	Try one of the **Extending** moves, *why do you think that?*
Problem	**Solution**
Other pupils may be tuning out as one pupil focuses deeper on his or her reasoning. How can the teacher help everyone deepen their own understanding?	Try the **Supporting** move revoicing and say, *‘who can put that into their own words?’*
Problem	**Solution**
The teacher needs to engage pupils beyond simply listening and repeating	Try the **extending** move, *do you agree or disagree…and why?*
Problem	**Solution**
How do we invite pupils in when we sense that the conversation is clear enough and there is enough common ground to move forward?	Try the **extending** move, *who can add on?*

Possible Resources

Counters	Counting Stick
Interlocking Cubes	Bead Strings
Coins	Number Lines
100 squares	99 squares
Place Value Arrow Cards	Base Ten/ Dienes Blocks
Empty Number Lines	Ten Frames
Interactive Whiteboard	Dice with various numbers of faces
Playing Cards	Calculators
Dominoes	Cuisenaire Rods
Digit Cards	Number Fans
Individual white boards	

Key Teaching Principles for Mental Maths[2]

- Encourage children to share their mental methods.
- Encourage children to choose efficient strategies.
- Encourage children to use informal jottings to keep track of the information they need when calculating.
- Commit regular time to teaching mental calculation strategies.
- Provide practice time with frequent opportunities for children to use one or more facts that they already know to work out more facts.
- Introduce practical approaches and jottings, with models and images children can use, to carry out calculations as they secure mental strategies.
- Encourage children in discussion when they explain their methods and strategies to you and their peers.
- Ensure that children can confidently add and subtract any pair of two-digit numbers mentally, using jottings to help them where necessary.

[2] Adapted from Crown (2010)

- Teach a mental strategy explicitly but in addition invite children to suggest an approach and to explain their methods of solution to the rest of the class.
- Hands on learning is important
- Provide suitable equipment for children to manipulate and explore how and why a calculation strategy works. That helps them to describe and visualise the method working.
- Encourage children to discuss their mistakes and difficulties in a positive way so that they learn from them.

Assessment

- A 'mental test' can help children to monitor changes in their own performance over time.

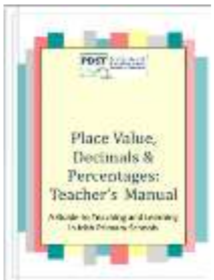
  The traditional mental arithmetic test involves a set of unseen questions. A worthwhile alternative is to give children examples of the type of questions 10 minutes in advance, so that they can think about the most efficient way to answer the questions. The purpose of this preparation time is not to try to commit answers to memory but to sort the questions into those they 'know' the answer to, and those that they need to figure out. Pairs of children can talk about their 'figuring out' methods and after the test the whole class can spend some time discussing the strategies they used. (Crown 2010)
- Collecting the questions, then giving children the test with the questions in a random order, also encourages attention to strategies. The same test can be used at a different time for children to try to beat their previous score. (Crown 2010)
- Don't use lengthy timed tests. Pupils get distracted by the pressure and abandon their reasoning strategies. They can lead to pupil anxiety, which does not support mathematical learning. If there is any purpose for a timed test of basic facts it may be for diagnosis – to determine which combinations are mastered and which remain to be learned. Even for diagnostic purposes timed tests should only occur once every couple of weeks. (Van De Walle, p.184)

Teacher Manuals for Supporting and Developing Mathematical Thinking

The instructional framework for supporting and developing pupils' mathematical thinking is described in detail in the following three PDST Manuals. Click on each image to access a free e-copy of the manuals.

- *Fractions: Teacher's Manual, A Guide to Teaching and Learning Fractions in Irish Primary Schools.*

- *Place Value, Decimals and Percentages: Teacher's Manual, A Guide to Teaching and Learning in Irish*



- *Shape and Space: Teacher's Manual, A Guide to Teaching and Learning in Irish Primary Schools*

Practical Activities for Developing Number Properties

Although mental maths is often used in the abstract, a solid foundation in number properties is central to pupils' success in developing and applying mental maths strategies. The following activities that explore the development of number properties are included as **pre-requisites for the development of mental maths strategies**.

Commutative Property

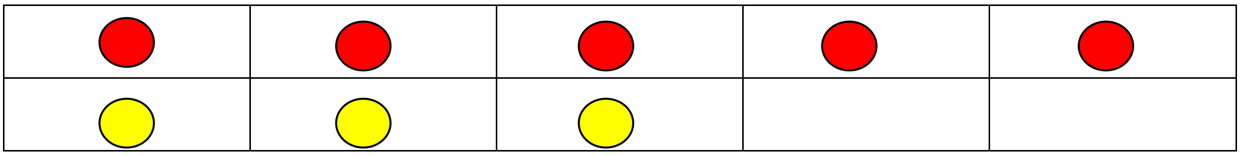
There are a number of concrete ways of constructing the commutative property of addition with pupils:

ICT Opportunities

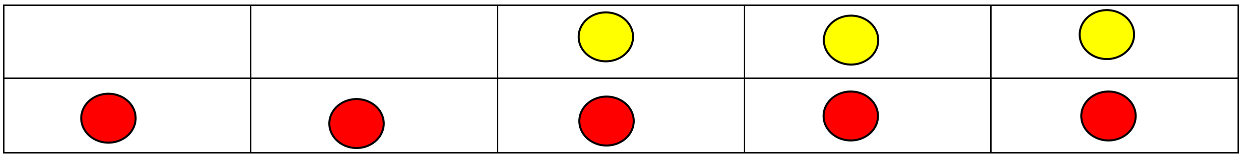
[Ten Frames]

Ten Frame

Children solve 5 + 3 using a ten frame.



If the ten frame is rotated 180 degrees, the problem is now 3 + 5



The answer remains the same.

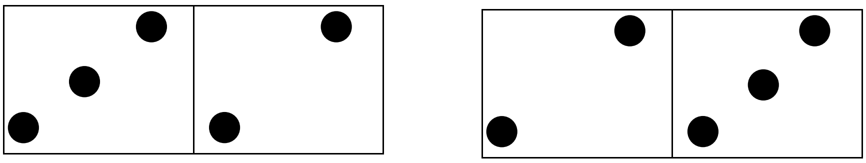
How could we use our Ten Frame to work out this sum?
What do you notice if you turn your ten frame upside down?
Did your answer change?
Let's try that with some other sums to check if it always works.
What does that tell us about the way we can add two numbers?
Do we always have to add the second number to the first number?
Record what you have learned in your learning log

Ten-Frames: The most common and perhaps most important model for relating numbers to five and ten is the ten-frame. The ten-frame is simply a 2 × 5 array in which counters or dots are placed to illustrate numbers.

Van de Walle p. 133

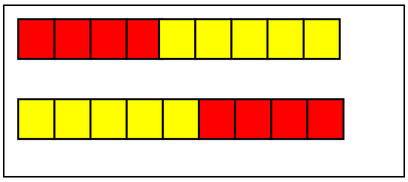
Flipping Dominoes

Children add both numbers on a domino. If the domino is flipped and the numbers are added in reverse order, what do you notice?



Cubes

Children combine sets of cubes to solve addition sums.



Clothes Hanger

Use a clothes hanger with three blue pegs on one side and two white pegs on the other side. 'Three and two altogether make five.' Now turn the hanger around. 'Two and three altogether make 5. Elicit from the pupils what they have noticed to draw their attention to the commutative property.

Ready, Set, Go Maths

Partitioning: through practical partitioning activities as advocated in the *Ready, Set, Go – Maths* programme, pupils will gain valuable experience of the commutative property.

In this example of partitioning of a set of 5, both facts of 5, 4 + 1 and its commutative equivalent, 1 + 4 are practically constructed by the pupils.

Associative Property

The above concrete activities to develop the commutative property can be adapted to develop the associative property of addition with pupils. Simply use addition sums involving three addends.

Adding Zero

> Occasionally pupils feel that 6 + 0 must be more than 6 because 'adding makes numbers bigger' or that 12 – 0 must be 11 because 'subtracting makes numbers smaller'. Instead of making arbitrary-sounding rules about adding and subtracting zero, build opportunities for discussing zero into the problem-solving routine.
>
> Van de Walle p. 154

Pose problems involving zero. For example, *Robert had eaten 8 grapes. He was too full to eat any more. How many did Robert eat altogether?* In discussion of the problem, use drawings/counters to illustrate the empty set (zero).

Practical Activities for Developing Addition Strategies

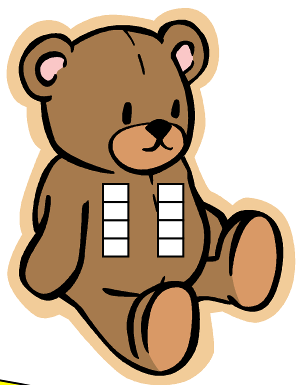
Counting forwards and backwards

- See PDST counting document and green section of *Ready, Set, Go – Maths.*

Doubles Strategy

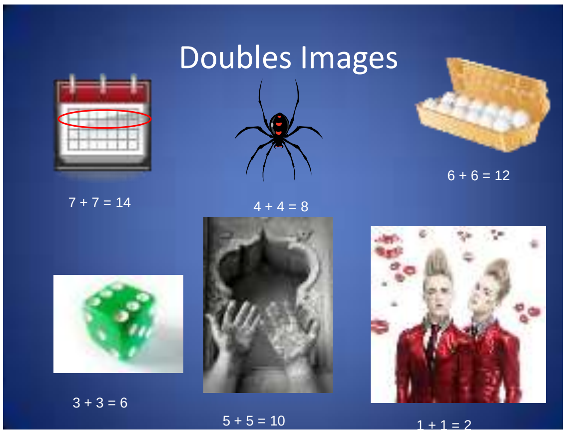
- Multi-link cubes: Use multi-link cubes to build up a visual picture of each double with the children.

- Doubles Templates: Use pictures of a double-decker bus with five windows on top and five windows on the bottom or a teddy bear wearing a coat with space for four buttons on his left and four buttons on his right.(Templates using any doubles combinations can be used.)Pupils use unifix cubes as passengers or buttons.



Let's put some buttons on Teddy. We can put three buttons on one side and three buttons on the other side. How many buttons does Teddy have now? Let's put passengers on the bus, five passengers upstairs and five passengers downstairs. How many passengers altogether?

- Doubles Images: Ask children to identify doubles in their environment and to record these in picture form, with the corresponding written double fact.[3]



Near Doubles Strategy

- Doubles/Near Doubles using ten frames

(A)

(B)

Ask the pupil, '*How many dots do you see? How do you see them?*'(PARRISH p. 110) Comparing ten frames (one a double, the other a near double) allows the child to establish the relationship between doubles and near doubles.

- A similar activity can be done, using multi-link cubes and doubles templates (see above) to develop the pupil's understanding of the relationship.

[3] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7^{th} edn. Pearson: P.17

Perhaps the most important strategy for pupils to know is the Make Ten strategy; or the combinations that make 10. Many of the basic addition facts can be solved using the Make 10 strategy.

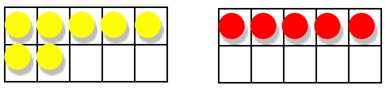
Facts of Ten

- Place counters on a ten frame and ask, ‘How many more to make ten?’ This activity can be repeated again and again until pupils have constructed all the combinations to make ten.

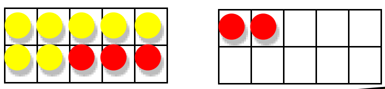
- Pupils use Cuisenaire Rods to make the Step Pattern for ten. See *Ready, Set, Go – Maths* p. 87 – 94 for guidance on using Cuisenaire rods.

Bridging Through Ten/Bridging a Multiple of Ten

- This strategy can be developed using two ten frames and counters.



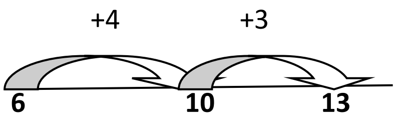
7 + 5



Let’s try adding 7 + 5 using our ten frames. Put 7 counters on your first frame. How many empty spaces do you have? Put 5 counters on the second frame. Now let’s add. When we add, we put the counters together. Do we have enough empty spaces for our 5 counters? We’ll fill those spaces up with counters from our second frame. How many do we have in our first frame now? And how many have we left on our second frame? Our sum was 7 + 5. What is it now? (10 + 2 = 12). Now try using two frames to work out 8 + 5, 7 + 4, 9 + 4, 9 + 6.

> In mental addition or subtraction, it is often useful to count on or back in two steps, bridging a multiple of 10. The empty number line, with multiples of 10 as 'landmarks', is helpful, since children can visualise jumping to them.
>
> Crown (2010), p.36

- Empty Number Lines [4]



6 + 7 can be worked out in two jumps on the empty number line, first a jump to 10, then to 13. The answer is the last point marked on the line, 13.

Reordering

This strategy is closely related to the commutative and associative properties of addition. It builds on the child's application of other strategies such as facts of ten, doubles, near-doubles.

- Present children with groups of three and then four numbers that they are to add together in their head. Make sure, that in each group of numbers, there are two numbers that have a total of 10. For example, 8 + 3 + 5 + 2. Discuss their methods. See if any children chose to add 8 + 2 first (facts of ten) and then add on the 5 + 3, or linked the 3 + 5 and added 8 (doubles)[5]

Explain how you added your numbers. Did anyone add the numbers in a different order? Why did you add those numbers together first? As well as reordering the numbers, what other strategy did you use? Peter, revoice what Sophie said.

[4] Crown (2010) *Teaching Children to Calculate Mentally*.p.36
[5] Crown (2010) *Teaching Children to Calculate Mentally*.p.32

Give pupils similar examples and encourage them to look for pairs of numbers that add to make ten or that make doubles before they start to add. Ask them to make up similar examples for each other.

- To assist pupils in expressing how they added a several numbers together, the following model will be useful: [6]

14 39 16 25 21

30 60

90

90 + 25 = 115

Partitioning by Place Value

Once students begin to understand place value, this is one of the first strategies they utilise. Each addend is broken into expanded form and like place value amounts are combined.119 + 126 can be added using this strategy as follows:

(100 + 10 + 9) + (100 + 20 + 6)

100 + 100 = 200

10 + 20 = 30

9 + 6 = 15

30 + 15 = 45

200 + 45 = 245

Parrish 2010, p.63, 164

- Developing Place Value concepts: See the PDST Manual *Place Value, Decimals and Percentages* http://www.pdst.ie/NumeracyMain for concrete, pictorial and abstract development of place value.

[6] Crown (2010) *Teaching Children to Calculate Mentally*.p.32

> Teaching Place Value is much more than requiring students to state how many hundreds, tens and ones are in a number or having them write a number in expanded notation. The true test of whether students understand place value is if they can apply their understanding in computation.
>
> Parrish, 2010 p.159

- Writing problems in a horizontal format encourages pupils to develop and use this strategy[7]

Compensating

> The compensating strategy is useful for adding and subtracting numbers that are close to a multiple of 10, such as numbers that end in 1 or 2, or 8 or 9. The number to be added or subtracted is rounded to a multiple of 10 plus or minus a small number. For example, adding 9 is carried out by adding 10, then subtracting 1; subtracting 18 is carried out by subtracting 20, then adding 2. A similar strategy works for adding or subtracting decimals that are close to whole numbers. For example, 1.4 + 2.9 = 1.4 + 3 – 0.1 or 2.45 – 1.9 = 2.45 – 2 + 0.1.
>
> Crown 2010, p. 40

- Compensation Game[8]: Prepare two sets of cards for a subtraction game. Set A has numbers from 12 to 27. Set B contains only 9 and 11 so that the game involves subtracting 9 and 11. Shuffle the cards and place them face down. Each child needs a playing board:

Playing Board

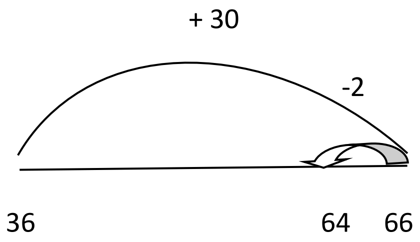
15	3	9	16
5	18	4	17
13	7	12	8
6	11	14	10
18	1	2	17

Pupils take turns choosing a number from set A and then one from set B. They subtract the number from set B from the one from set A and mark the answer on their board. The first person to get three numbers

[7] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies*
[8] Crown (2010) *Teaching Children to Calculate Mentally*.p.41

in a row on the board wins. Discuss the strategy used by pupils to subtract efficiently. Encourage the use of compensating. (Crown p. 41)

- Hundred Square[9]: The hundred square is useful for adding and subtracting tens and numbers close to 10. To find 36 + 28, first find 36 + 30 by going down three rows, then compensate by going back along that row two places. Subtracting 10 is modelled by moving to numbers in the row above. (Crown p. 41)
- Empty Number Line[10]: Use a number line to support mental calculations such as 36 + 28 by counting on 30 and compensating by counting back 2.



The goal of compensation is to manipulate the numbers into easier, friendly numbers to add. When compensating, students will remove a specific amount from one addend and give that exact amount to the other addend to make friendlier numbers. Taking from one addend and giving the same quantity to the other added to maintain the total sum is a big mathematical idea in addition. This strategy will often begin to emerge...as a way to make doubles and tens.

Parrish, 2010 p.62

[9] Crown (2010) *Teaching Children to Calculate Mentally*.p.41
[10] Crown (2010) *Teaching Children to Calculate Mentally*.p.41

Bridging Through 60 (to calculate a time interval)[11]

A digital clock displaying 9:59 will, in two minutes time, read 10:01 not 9:61. When children use minutes and hours to calculate time intervals, they have to bridge through 60. So to find the time 20 minutes after 8:50am, for example, children might say 8:50 plus 10 minutes takes us to 9:00am, then add another 10 minutes.

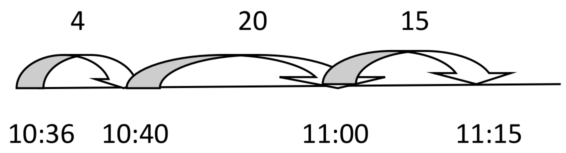
Crown, 2010 p. 45

- Use a digital clock in the classroom[12].

10:36

Get the class to look at it at various times of the day and ask: ‘How many minutes is it to the next hour (or next o’clock) Encourage children to count on from 36 to 40, then to 50, then to 60, to give a total of 24 minutes. Then ask questions such as: ‘How long will it be to 11:15? Get them to count on to 11:00 and then add on the extra 15 minutes.

The calculation can be modelled on a number line labelled in hours and minutes.



Some children may think that minutes on a digital clock behave like ordinary numbers, so that they might count on 59, 60, 61 and so on, not realising that at 60 the numbers revert to zero as the hour is reached. It helps if you draw attention to what happens to the clock soon after, say, 9:58 and stress the difference between this and other digital meters such as electricity meters or car odometer.

Crown, 2010 p.46

[11] Crown (2010) *Teaching Children to Calculate Mentally*.p.45
[12] Crown (2010) *Teaching Children to Calculate Mentally*.p.46

- Give a group of children statements such as[13]:

  'Jane leaves home at 8:35am. She arrives at school at 9:10am. How long is her journey?'
  Elicit from the children their methods of finding the answer, writing each on the board. Some may say: '8:35, 8:40, 8:50, 9:00, 9:10,' counting 5 and 10 and 10 and 10 to give the total time. Other may say '8:35 and 25 minutes takes us to 9:00, so add on another 10 minutes'

Children need to remember that, for minutes, they need to count up to 60 before getting to the next hour. Some children might be tempted to say 8:35, 8:40, 8:50, 8:60, and so on, expecting to go on until they get to 100. Referring to a clock face should help them to see why this is incorrect.

Crown, 2010 p.46

- Use local bus and train timetables, asking questions such as, 'How long does the 8:30 train take to get to Dublin etc?' Encourage children to build the starting times up to the next hour, and then add on the remaining minutes.[14]

- Plan a journey using information from a bus or train timetable. Discuss the strategies that children use to find the times. Encourage them to use empty number lines to model their calculations.

[13] Crown (2010) *Teaching Children to Calculate Mentally*.p.46
[14] Crown (2010) *Teaching Children to Calculate Mentally*.p.47

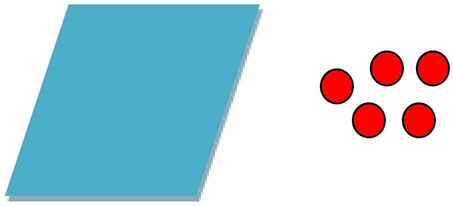
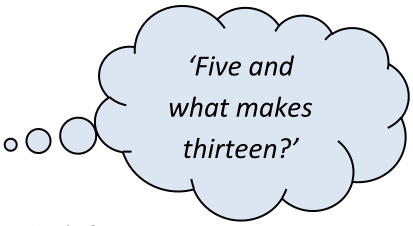
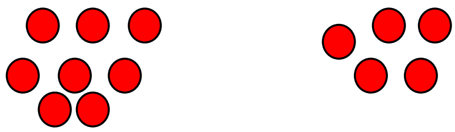
Practical Activities for Developing Subtraction Strategies

Subtraction as 'Think - Addition'

Evidence suggests that children learn very few, if any, subtraction facts without first mastering the corresponding addition facts. In other words, mastery of 3 + 5 can be thought of as prerequisite knowledge for learning the facts 8 – 3 and 8 – 5.
Without opportunities to learn and use reasoning strategies, students may continue to rely on counting strategies to come up with subtraction facts, a slow and often inaccurate approach. When children see 9 – 4, you want them to think spontaneously. 'Four and *what* makes nine?'

Van de Walle p. 175

- Connecting Subtraction to Addition Knowledge: [15]

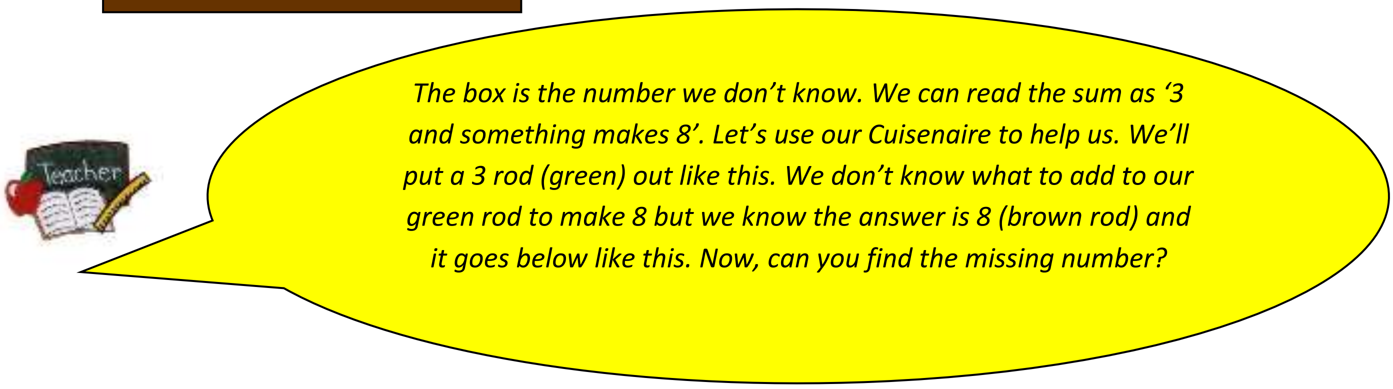
1. Count out 13 counters and cover. 	2. Count and remove 5. Keep these in view 
3. Think:  8! 8 left. 13 – 5 = 8	4. Uncover 

Try this model with other numbers to provide opportunities for children to connect their addition knowledge to subtraction.

[15] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally*. 7th edn. P.175

- Box Numbers Game:[16] Put out a set of objects (counters, teddies, cubes etc.) Ask the pupil how many there are. The pupil closes their eyes. Hide some of the objects under a box. Ask the pupil. 'How many were there?', 'How many can you see?', 'So how many are under the box?' Continue the game hiding a different amount each time.

- Box Addition using Cuisenaire Rods: [17]The pupils use Cuisenaire Rods to solve the following type of problem (box numbers): $3 + \boxed{} = 8$



- Typical problems that require the use of the subtraction as 'Think-Addition' strategy[18]:

  How many can sit in the back of my car? There's room for 2 in the front and it holds 5 passengers altogether.

  There were 4 ducks in the pond. When I came back I saw 8. What had happened?

  I had 7 €1 coins in my purse. How many would I have to add to make it up to €10?

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[16] Pitt, E (2010) *Ready Set Go Maths: A practical handbook for teachers* p. 67
[17] Pitt, E (2010) *Ready Set Go Maths: A practical handbook for teachers* p.94
[18] Pitt, E (2010) *Ready Set Go Maths: A practical handbook for teachers* p.94

Keeping a Constant Difference

As students begin to understand subtraction as the difference between two quantities, they can investigate what occurs if both numbers are changed by the same amount. Allowing students to explore this relationship with smaller problems such as 5 – 3 is a way to help them build this understanding. If 5 and 3 are both changed by + 2, the problem 7 – 5 will result. Notice that there is still a difference of 2. What if we removed 2 from each number in the problem 5 – 3? We would then create the problem 3 – 1, which still results in a difference of 2. Adding or subtracting the same quantity from both the subtrahend and minuend maintains the difference between the numbers. Manipulating the numbers in this way allows the student to create a friendlier problem without compromising the result.

Parrish, 2010 p.178

- Examining the distance between two numbers on a number line or metre stick helps students to understand the logic of this strategy. [19]

- Provide pupils with a selection of problems that can be solved efficiently using this strategy. Encourage the pupils to alter both numbers by the same amount to make friendlier numbers that can be subtracted more easily. Ask pupils to share their approaches.

51 – 26	13 – 9	39 – 17	61 – 29	100 – 51
200 – 91	164 – 119	114 – 89	391 – 146	86 – 47

Towards Fluency in Applying Strategies in addition to Mastery of Facts

Doubles

- Pose word problems. For example, Sarah and Tom bought 7 sweets each. How many sweets did they have altogether?
- Calculator Doubles[20]:Enter 2 x = Let one child say a double fact e.g. 6 plus 6. The child with the calculator should press 6, try to give the double (12) and then press = to see the correct double on display.

[19] Prince Edward Island, Canada, Dept of Education (2008) Mental Math Fact Learning, Mental Computation, Estimation, Grade 3, Teacher's Guide

Near Doubles

- Pose word problems. For example, I had €8 in my money box. I saved €9 more. How much money do I have now?

- Double Dice Plus One:[21]

  The child rolls a dice labelled 4-9. They say the sum of the number shown plus the number that is one greater. E.g. For 8: 'Eight plus nine is 17'

- Play 'Think of a number'[22].

  Use a rule than involves doubling and adding or subtracting a small number. For example, *'I'm thinking of a number. I doubled it and added 3. My answer is 43. What was my number?'*

- Triple Dice Game:[23]

  This game for a group of children needs three dice. One is numbered 1 to 6; a second has four faces marked with a D for 'double' and two blank faces; the third is marked +1, +1, +1, +2, - 1, - 1.
  Children take turns to throw the three dice and record the outcome. They then decide what number to make. For example, if they throw 3, D and + 1 they could: double the 3 then add 1 to make 7, or they could add 1 to 3 to make 4 and then double 4 to make 8.
  What is the smallest possible total?
  What is the largest possible total?
  What totals are possible with these three dice? Which totals can be made in the most ways?
  Encourage children to reflect on the processes rather than to find just one answer. (Crown 2010, p.44)

- Consecutive Numbers: [24]

  Get children to practise adding consecutive numbers such as 45 and 46. Then give children statements such as: 'I add two consecutive numbers and the total is 63'. Ask them: 'What numbers did I add?'

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[20] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn p. 173
[21] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (201o). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn
[22] Crown (2010) *Teaching Children to Calculate Mentally*.p.44
[23] Crown (2010) *Teaching Children to Calculate Mentally*.p.44
[24] Crown (2010) *Teaching Children to Calculate Mentally*.p.45

‘Think of a number’ activities require children to ‘undo’ a process by using inverse operations. This activity gives practice in both halving and doubling. Invite children to invent similar examples themselves.

Crown, 2010 p.44

Facts of Ten

- Play the ‘Tens’ Card Game using a deck of cards with picture cards removed. Ace is worth 1

<u>Instructions:</u>

This is a game for two players. The winner is the first person to get rid of all of their cards, by adding pairs of numbers that make ten. Deal ten cards to each player. The remaining cards are left in a pile, face down, in the middle of the table. To begin, each player removes any pairs of their cards that add up to 10, for example, 9 and ace, 4 and 6 etc. from their own ten cards. They state the ten fact and place the cards face up on the table, so their opponent can see them. With their remaining cards they must ask their opponent for cards that will complete their pairs to make 10. For example, if they are left with a 2, 7 and 6, at each turn they will ask their opponent for either an 8, 3 or 4. If their opponent does not have the card the player must take a card from the pile on the table. If their opponent does have the required card, they give it to the other player who then places the pair of cards making ten, face up, on the table, stating the ten fact. The winner is the first person who gets rid of all their cards.

- Say the 10 Fact [25]
  Hold up a ten-frame card, and have the children say the ’10 fact’. For a card with seven dots, the response is ‘seven and three is ten.’ Later, with a blank ten-frame drawn on the board, say a number less than ten. Children start with that number and complete the ’10 fact’. If you say, ‘four’, they say, ‘four and six is ten’.

- Reordering Game:[26]
  Have regular short, brisk practice sessions where children are given 10 questions such as: 2 + 7 + 8 + 5 + 4 + 3 where some pairs total 10. Encourage children to time their responses, keep a personal record of their times and try to beat their personal best.
  Give children the same set of questions at regular intervals and encourage them to see how rapidly they can get the answers. This should ensure that every child sees that they have made progress.

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[25] [25] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn p. 172

[26] Crown (2010) *Teaching Children to Calculate Mentally*.p.32

Consolidation Activities

Strategy Cards

This activity provides practice in identifying and using strategies to find out answers to number facts.

Materials needed:

- Assorted number fact cards, for example 5 + 6 =
- Three strategy labels, for example,' Doubles', 'Near Doubles', ' Make 10'.

Pupils take turns selecting a number fact card and placing it under the correct label. For example, 5 + 6 would go under 'Near Doubles'. Pupils can then either say the answer; use the strategy to find the answer or another child could provide the answer.

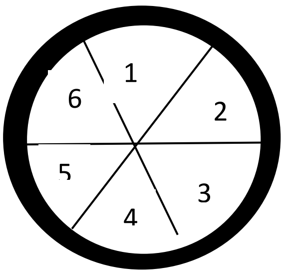
Adding Snap

Materials needed:

- Deck of playing cards. Ace is worth 1, picture cards are worth 10,
- *Number of Players:* 3 (one acts as referee)

Divide all the cards equally between both players. The referee calls counts down '3, 2, 1, Turn!' and both players turn over one card each and place it on the table. The first player to add both cards correctly and call out the answer gets the cards. For example, King + 3 = 13. The winner is the first person to win all the cards or the person with most cards after a period of time. The referee ensures fairness in relation to cards being placed on time by both players. S/he decides if cards need to be replayed in the case of both players calling the answer out at the same time. The referee also ensures the accuracy of answers. *Variation*: Teacher can intervene at any stage and ask a player to explain the strategy they used to get the answer. If they can explain, they win a card from their opponent.

Dartboard[27]



Three darts land on this board. Darts in the outside ring score double that number. More than one dart can land in an area. Find different ways of scoring 26. How many different ways can you find? (Crown 2010 p. 48)

[27] Crown (2010) *Teaching Children to Calculate Mentally*.p.48

Closest to 100[28]

You need a set of digit cards from 1 to 7. Arrange your cards with + signs in between them. Use each card once. How close can you get to 100? Here is an example:

52+13+46+7 = 118.

Can you get closer to 100?

Magic Square Gone Wrong[29]

23	10	17	4	11
6	18	5	12	24
19	1	13	25	7
2	16	21	8	20
15	22	9	16	3

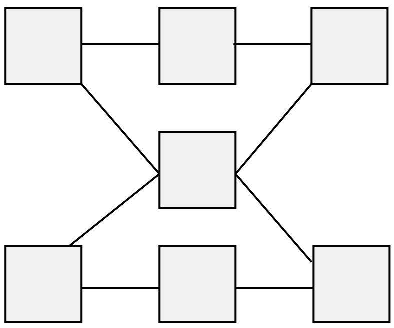
This is a magic square. The numbers in any row, column or the two diagonals have the same total. Unfortunately, there is something wrong. One of the numbers is incorrect. Which number is it? What should it be? (The number 16 towards the bottom left corner should in fact be 14)

Digit Card Game[30]

Use digit cards 1, 2,3,4,5,6,7,8. Put these eight cards in three groups. There must be at least one card in each group. In each group, the sum of the numbers of the cards must be the same. Find three different ways to do it. (The eight cards have a total of 36, so the total in each group must be 12. The three ways are: 8,4 7,5 6,3,2,1 8,3,1 7,5 6,4,2 8,4 7,3,2 6,5,1

Number Tree[31]

You need seven number cards 1,2,3,4,5,6,7. Arrange the cards on this grid. Each line of three numbers must add up to 12.



Can you find two other ways to do it? (Put 4 in the centre and the rest fall into place)

[28] Crown (2010) *Teaching Children to Calculate Mentally*.p.48
[29] Crown (2010) *Teaching Children to Calculate Mentally*.p.49

[30] Crown (2010) *Teaching Children to Calculate Mentally*.p.49
[31] Crown (2010) *Teaching Children to Calculate Mentally*.p.50

Fill the Grid[32]

This is a game for two players with three dice. Each of you should draw a grid like this.

Take turns to roll all three dice. Write the total score on your own grid. When the grids are full, keep rolling. This time, if the total score is on either player's gird, cross it out. The winner is the first player to get all their numbers crossed out. (Crown p.50)

If You Didn't Know[33]...

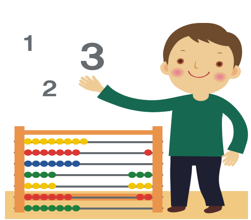
Pose the following question to the pupils: if you did not know the answer to 8 + 5 (or any fact that you want pupils to think about), what are some really good ways to get the answer? Explain that 'really good' means that you don't have to count and you can do it in your head. Use a Think-Pair-Share approach in which pupils discuss their ideas with a partner before they share them with the class.

Loop Games

Loop games can be an excellent, enjoyable way to support pupils in mastering basic number facts. A variety of loop game cards can be downloaded here
http://www.primaryresources.co.uk/maths/mathsA2.htm

[32] Crown (2010) *Teaching Children to Calculate Mentally*.p.50

[33] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally*. 7th edn p. 174



Counting in the Junior Classes

Adapted from Ready Set Go Maths (E. Pitt)

The ability to recite number words in order is a prerequisite to developing the ability of counting a wide range of objects. Pupils' ability to recite numbers in order is usually more developed than their ability to count objects. However, pupils who find mathematics difficult may not have a full grasp of number sequences. Providing daily 10 minute oral counting activities can increase pupils' confidence in this area.

These sessions should have:

- A lively pace
- Enthusiastic participation
- Two or three different short focussed activities (variety will maintain interest)
- Physical activity
- Choral response
- Individual response

Pupils should develop the ability to:

- Count forwards
- Count backwards
- Count forwards/backwards from different starting points.

Some suggested activities include:

Counting Stick

Rhythm Counting

Use actions such as: clapping, slapping, tapping. Pupils chant number words in time with the rhythm.

Counting Apple (pendulum)

Pupils chant numbers in time to a swinging apple (weight attached to a long string). This can also be used for counting quantities/sets of objects.

Live Number Line

Pupils are given large cards with each number and are asked to line up in order of the sequence. Teacher/pupil then asks other pupils to swop with those in the line emphasising language: before/ after, more than /less than, between, first/second..., Largest/smallest, etc.

The Sound of a Number Game (also known as Counting Can)

Teacher drops cubes into a tin. Pupils close their eyes and count silently in their heads. When pupils become proficient in this, add some cubes to the tin, but pause for 2 seconds, then continue adding cubes to the tin. Ask the pupils how many cubes are now in the tin. This promotes the idea of 'counting on'. It can be challenging for pupils to retain the first number in their heads and then count on from it. This can be extended by removing some cubes and asking the pupils: 'How many were in the tin?', 'How many are there now?', so 'How many must be in my hand?'

Stand and Sit

Pupils stand and then sit while saying the number sequence required, for example, counting forwards or backwards from a certain number.

Clap and Snap

Count forwards clapping in time, then count backwards snapping fingers in time.

Stamp and Tap

Pupils find a space facing the board. Count forwards stamping feet in time. Stop at required number word and turn in opposite direction. Now count back tapping their shoulders in time. (Do this without pausing!)

Class Number Line (Pegs on a line)

Count forwards/backwards while looking at each number. Teacher points to a number and pupils say that number together. Say number before/ after given number. Turn one number around, pupils tell (individually) hidden number and explain their thinking.

Show me

Teacher shows flash cards with different numbers of objects. Pupils count silently and show corresponding number using digit cards or number fans.

Move your marker

Pupils have number line (1-5/1-10/100 square) and a counter/cube. Teacher gives instructions, for example, "Put your counter on the number that comes just before/after" or on any number greater than…. /between, etc.

Head and shoulders

Tap head and shoulders in turn to a rhythm. Say number sequence while doing this. Then develop it asking pupils to only say the number on the head tap. Number on shoulder tap is said silently.

Pass the Teddy

Pupils stand in a circle. As Teddy is passed around the ring pupils say the next number or can say "I am one I pass it to two", etc.

Count Around

Pupils stand in a circle and count around, each child saying the next number in the sequence. Start counting at one, pupil who says number 12 sits down. Keep going until only one pupil is standing. (This can be varied using shorter/longer sequences, using different starting/finishing points, doing it backwards, etc.)

Counting Choir

Divide class into 3 groups. Teacher is in the role of conductor, with a baton. Teacher begins to count and then points the baton at one group to continue counting in unison. Teacher then points to different group and continues with the count.

Hand Game

Teacher picks a starting point, for example 12. If teacher raises her/his hand up it means count one digit more, if the hand faces down it means one digit less. This can be extended by the teacher directing his/her hand to the right (add 10 more) and to the left (10 less).

100 Square Jigsaw

A 100 square cut up into segments. Teacher hands out the segments to the children. The pupils put the 100 square back together.

Guess my Number

Show the pupils a section of a digit and get them to guess what the number is. Ask them to justify their answer.

Find and Press

Each pupil should have a calculator. Always begin by having the pupils press the clear key. Then you say a number, and the pupils press that number on the calculator. If you have an interactive whiteboard calculator, you can then show the pupils the correct key so that they can confirm their responses, or you can write the number on the board for pupils to check. Begin with single-digit numbers. Later, progress to two or three numbers called in succession. For example, call, ‘Three, seven, one.’ Pupils press the complete string of numbers as called.

Counting On with Counters

Give each pupil a collection of 10/12 small counters and get pupils to line the counters up from left to right on their desks. Tell them to count four counters and push them under their left hands or place them in a cup. Then say, ‘Point to your hand. How many are there?’ (4). ‘So, let’s count like this: f-o-u-r (pointing to their hand), five, six...’ Repeat with other numbers under the hand.

Make a Two-More-Than Set

Provide pupils with about six dot cards. Their task is to construct a set of counters that is two more than the set shown on the card. Similarly, spread out 8-10 dot cards, and find another card for each that is two less than the card shown. (Omit the 1 and 2 cards for two less than, and so on.)

A Calculator Two-More-Two Less Than Machine

Teach pupils how to make a two-more-than machine. Press 0 +2=. This makes the calculator a two-more-than machine. Now press any number for example 5. Pupils hold their finger over the = key and predict the number that is two more than 5. Then they press = to confirm. If they do not press any of the operation keys (+, -, x, ÷), the ‘machine’ will continue to perform in this way.

References

Baroody, A (2006) *Why Children Have Difficulties Mastering the Basic Number Combinations and how to help them.* The National Council of Teachers of Mathematics.

Chapin, O'Connor and Anderson (2013) *Classroom Discussions In Math: A Teacher's Guide for Using Talk Moves to Support the Common Core and More.* Math Solutions

Crown (2010) *Teaching Children to Calculate Mentally*. Publications.education.gov.uk

Deboys, M. & Pitt, E. (1979). *Lines of Development in Primary Mathematics.* Queens University: Belfast.

Prince Edward Island, Canada, Dept of Education (2008) *Mental Math Fact Learning, Mental Computation, Estimation, Grade 3, Teacher's Guide*

Fraivillg, J. L., Murphy, L. A. & Fuson, K. C. (1999). Advancing pupils' mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education*, *30*(2) 148-170.

National Council for Curriculum and Assessment (1999). *Primary School Mathematics Curriculum*. Dublin: Stationery Office.

National Council for Curriculum and Assessment (1999). *Primary School Mathematics Curriculum: Teacher Guidelines*. Dublin: Stationery Office.

Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* Scholastic

PCSP *Making Tables Add Up-Part 1*acccessed via http://www.ppds.ie/pcsparchive/

PCSP *Making Tables Add Up Again* accessed via http://www.ppds.ie/pcsparchive/

Pitt, E (2010) *Ready Set Go Maths: A practical handbook for teachers*. Council for the Curriculum, Examinations and Assessment (CCEA) in Belfast.

Suggate, J., Davis, A. & Goulding, M (2010). *Mathematical Knowledge for Primary Teachers*. 4th edn. Oxford: Routledge.

Van de Walle, J. (2007). *Elementary and Middle School Mathematics Teaching Developmentally*. 6th edn. Pearson: Allyn and Bacon.

Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). *Elementary and Middle School Mathematics Teaching Developmentally.* 8th edn. Pearson: Allyn and Bacon.