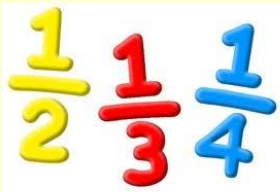


Fractions: Teacher's Manual

A Guide to Teaching and Learning Fractions in Irish Primary Schools

This manual has been designed by members of the Professional Development Service for Teachers. Its sole purpose is to enhance teaching and learning in Irish primary schools and will be mediated to practising teachers in the professional development setting. Thereafter it will be available as a free downloadable resource on www.pdst.ie for use in the classroom. This resource is strictly the intellectual property of PDST and it is not intended that it be made commercially available through publishers. All ideas, suggestions and activities remain the intellectual property of the authors (all ideas and activities that were sourced elsewhere and are not those of the authors are acknowledged throughout the manual).

It is not permitted to use this manual for any purpose other than as a resource to enhance teaching and learning. Any queries related to its usage should be sent in writing to:

Professional Development Service for Teachers,

14, Joyce Way,

Park West Business Park,

Nangor Road,

Dublin 12.

Contents

Aim of the Guide

The aim of this resource is to assist teachers in teaching the strand unit of Fractions (1st to 6th class). This strand unit is not applicable for infant classes. The resource is intended to complement and support the implementation of the Primary School Mathematics Curriculum (PSMC) rather than replace it. By providing additional guidance in the teaching and learning of fractions, this resource attempts to illuminate a pedagogical framework for enhancing mathematical thinking, that is, methods of eliciting, supporting and extending higher-order mathematics skills such as reasoning; communicating and expressing; integrating and connecting; and applying and problem solving.

Possible Resources

The following resources may be useful in developing and consolidating a number of concepts in fractions. This is not an exhaustive list and other resources may also be useful.

dienes blocks (base 10 materials)	**dice**
cuisenaire rods	**fraction, decimal, percentage walls**
fraction bars	**pie fraction sets**
class number lines (clothes line and pegs style), table top number lines	**playing cards**
empty number lines	**dominoes**
counting sticks	**notation/transition boards**
5 frame, 10 frame	**place value chart/template**
calendar	**calculators**
number line (with and without numbers)	**abacus**
hundred square (with and without numbers)	**food: hula hoops, dolly mixtures, smarties, cakes, pizzas, tortillas…**
number fans	**hundredths disc**
number balance	**10 x 10 grid paper**
digit cards	**dotted paper**

Differentiation

The approach taken to factions in this manual which uses three different models for exploring each concept, lends itself ideally to differentiated teaching and learning. For this reason the approach is ideal for use in the learning support, resource and special class settings.

The area model may appeal to spatial learners.

The linear model is compatible to the logical or the spatial learner.

Finally the set model allows for tangible and kinaesthetic learning experiences.

All models allow for the use of manipulatives and concrete materials and transfer to both the pictorial and the abstract representations. This myriad of learning experiences for the development of the same concept means that different learning styles and abilities are catered for as well as providing repeated opportunities to consolidate learning in a fun and interactive way. The learning trajectory is incremental as are the three stages of concrete, pictorial and abstract. As in all good teaching and learning environments the child dictates their starting point and the rate at which they move along the trajectory. Teachers in the multi class context will find the trajectory very helpful in this regard. Finally the instructional framework advocates a differentiated approach to questioning as a fundamental mode of assessment. Examples of various levels of questioning are evident throughout the activities in this manual.

Linkage

Although this guide focuses on one strand unit (fractions) of one strand (number), it is intended that the links to other strands, strand units and subjects would be made where applicable. Some examples of the possible linkage of fractions within the maths curriculum can be seen in Table 1.1 for first and second class, Table 1.2 for third and fourth class, and Table 1.3 for fifth and sixth class.

Table 1.1 Possible linkage of fractions across the maths curriculum (first and second classes)

Class Level	Strand	Strand Unit	Objective
First & **Second**	Number	Fractions	• Establish and identify half **(and quarters)** of sets to 20

Class Level	Strand	Strand Unit	Objective
First & Second	Number	Counting and Numeration	• Estimate the number of objects in a set 0–20 • Compare equivalent and non-equivalent sets 0-20 • Order sets of objects by number • Use the language of ordinal number, first to tenth
First & **Second**	Number	Number Operations	• Explore and discuss repeated addition and group counting • Solve one-step **(and two-step)** problems involving addition and subtraction
First & **Second**	Shape & Space	2-D Shapes	• Construct and draw 2-D shapes • Combine and partition 2-D shapes • Identify halves **(and quarters)** of 2-D shapes
First & Second	Shape & Space	3-D Shapes	• Solve and complete practical tasks and problems involving 2-D and 3-D shapes
Second	Shape & Space	Symmetry	• Identify line symmetry in shapes and in the environment
First & Second	Measures	Length	• Solve and complete practical tasks and problems involving length
Second	Measures	Capacity	• Estimate, measure and record capacity using litre, half-litre, and quarter-litre bottles and solve simple problems
First & **Second**	Measures	Time	• Read time in hours and half-hours on 12-hour analogue clock **(and digital clock)**

Table 1.2 Possible linkage of fractions across the maths curriculum (third and fourth classes)

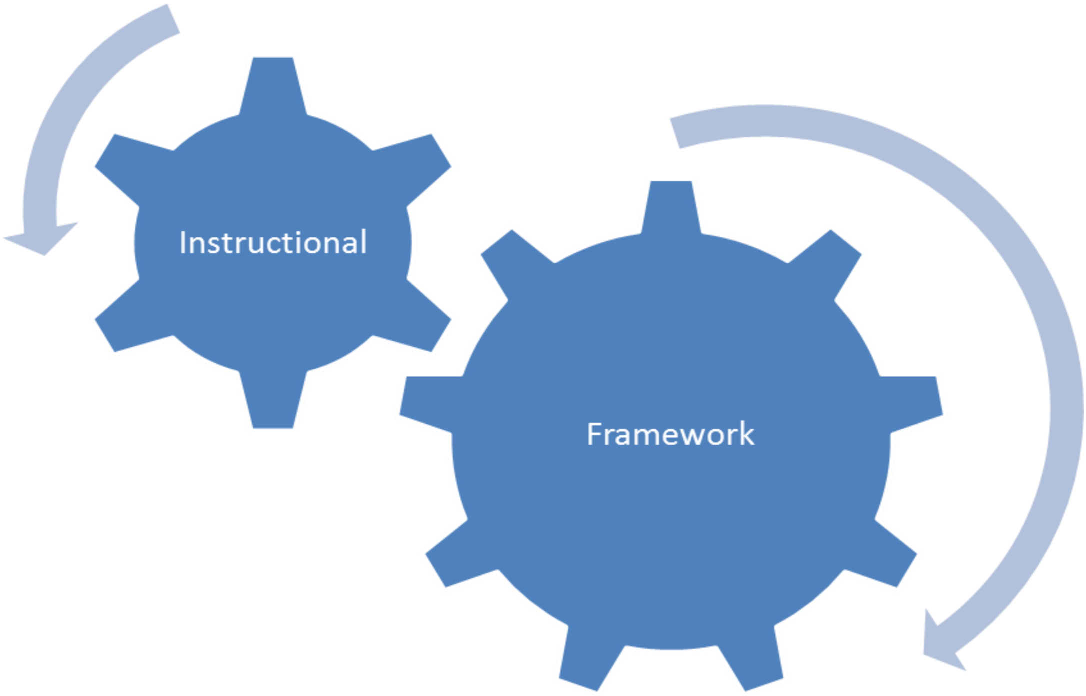
Class Level	**Strand**	**Strand Unit**	**Objective**
Third & **Fourth**	Number	Fractions	• Identify fractions and equivalent forms of fractions with denominators 2, 4, 8, 10 **(3, 5, 6, 9, 12)** • Compare and order fractions with appropriate denominators and position on the number line • Calculate a fraction of a set using concrete materials • Develop an understanding of the relationship between fractions and division • Calculate a unit fraction of a number and calculate a number, given a unit fraction of a number • Solve and complete practical tasks and problems involving fractions • **Express one number as a fraction of another number**

Class Level	**Strand**	**Strand Unit**	**Objective**
Third & **Fourth**	Number	Multiplication	• Develop an understanding of multiplication as repeated addition and vice versa • Explore, understand and apply the zero, commutative and distributive **(and associative)** properties of multiplication • Develop and/or recall multiplication facts within 100 • Solve and complete practical tasks and problems involving multiplication of whole numbers
Third & Fourth	Number	Division	• Develop an understanding of division as sharing and repeated subtraction, without and with remainders • Develop and/or recall division facts within 100
Third & **Fourth**	Number	Decimals	• Identify tenths and express in decimal form • **Express tenths and hundredths as fractions and decimals**
Third & Fourth	Shape & Space	2-D Shapes	• Construct and draw 2-D shapes • Solve and complete practical tasks and problems involving 2-D shapes
Third & **Fourth**	Shape & Space	3-D Shapes	• Identify and draw lines of symmetry in two-dimensional shapes • **Use understanding of line symmetry to complete missing half of a shape, picture or pattern**
Fourth	Measures	Length	• **Rename units of length using decimal or fraction form**
Fourth	Measures	Weight	• **Rename units of weight using decimal or fraction form**
Fourth	Measures	Capacity	• **Rename units of capacity using decimal and fraction form**
Fourth	Data	Representing and Interpreting Data	• **Read and interpret bar-line graphs and simple pie charts involving use of** $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{3}$

Table 1.3 Possible linkage of fractions across the maths curriculum (fifth and sixth classes)

Class Level	**Strand**	**Strand Unit**	**Objective**
Fifth & **Sixth**	Number	Fractions	• Compare and order fractions and identify equivalent forms of fractions with denominators 2-12 • Express improper fractions as mixed numbers and vice versa and position them on the number line • Add and subtract simple fractions and mixed numbers • Multiply a fraction by a fraction **(by a whole number)** • Express tenths, hundredths and thousandths in both fractional and decimal form • **Divide a whole number by a unit fraction** • **Understand and use simple ratio**

Class Level	**Strand**	**Strand Unit**	**Objective**
Fifth & Sixth	Number	Decimals and Percentages	• Develop an understanding of simple percentages and relate them back to fractions and decimals • Compare and order fractions and decimals • Solve problems involving operations with whole numbers, fractions, decimals and simple percentages
Fifth & **Sixth**	Number	Number Theory	• Identify **(common)** factors and multiples • Develop and/or recall division facts within 100
Fifth & **Sixth**	Shape & Space	2-D Shapes	• Classify 2-D shapes according to their lines of symmetry • Use 2-D shapes and properties to solve problems
Sixth	Measures	Weight	• **Rename measures of weight (express results as fractions or decimals of appropriate metric units)**
Sixth	Measures	Capacity	• **Rename measures of capacity (express results as fractions or decimals of appropriate metric units)**
Fifth & **Sixth**	Data	Representing and Interpreting Data	• Collect, organise and represent data using pictograms, single and multiple bar charts and simple pie charts **(using pie charts and trend graphs)** • Read and interpret pictograms, single and multiple bar charts, and pie charts **(pie charts and trend graphs)** • Compile and use simple data sets • Use data sets to solve problems



Instructional Strategies

Table 1.4 on the following page illustrates a framework for advancing mathematical thinking. Although it does not explicitly refer to concrete materials or manipulatives, the use of these are often a prerequisite for developing mathematical thinking and can be used as a stimulus for this type of classroom discourse.

Table 1.4 Strategies for Supporting and Developing Mathematical Thinking

Eliciting	Supporting	Extending
Facilitates pupils' responding Elicits many solution methods for one problem from the entire class *e.g. "Who did it another way?; did anyone do it differently?; did someone do it in a different way to X?; is there another way of doing it?"* Waits for pupils' descriptions of solution methods and encourages elaboration Creates a safe environment for mathematical thinking *e.g. all efforts are valued and errors are used as learning points* Promotes collaborative problem solving ***Orchestrates classroom discussions*** Uses pupils explanations for lesson's content Identifies ideas and methods that need to be shared publicly *e.g. "John could you share your method with all of us; Mary has an interesting idea which I think would be useful for us to hear."*	***Supports describer's thinking*** Reminds pupils of conceptually similar problem situations Directs group help for an individual student through collective group responsibility Assists individual pupils in clarifying their own solution methods ***Supports listeners' thinking*** Provides teacher-led instant replays *e.g. "Harry suggests that ...; So what you did was ...; So you think that ...".* Demonstrates teacher-selected solution methods without endorsing the adoption of a particular method *e.g. "I have an idea ...; How about ...?; Would it work if we ...?; Could we ...?".* ***Supports describer's and listeners' thinking*** Records representation of each solution method on the board Asks a different student to explain a peer's method *e.g. revoicing (see footnote on page 8)*	***Maintains high standards and expectations for all pupils*** Asks all pupils to attempt to solve difficult problems and to try various solution methods ***Encourages mathematical reflection*** Facilitates development of mathematical skills as outlined in the PSMC for each class level *e.g. reasoning, hypothesising, justifying, etc.* Promotes use of learning logs by all pupils *e.g. see Appendix A for a sample learning log* ***Goes beyond initial solution methods*** Pushes individual pupils to try alternative solution methods for one problem situation Encourages pupils to critically analyse and evaluate solution methods *e.g. by asking themselves "are there other ways of solving this?; which is the most efficient way?; which way is easiest to understand and why?".* Encourages pupils to articulate, justify and refine mathematical thinking *Revoicing can also be used here* Uses pupils' responses, questions, and problems as core lesson including student-generated problems ***Cultivates love of challenge***

This is adapted from Fraivillig, Murphy and Fuson's (1999) Advancing Pupils' Mathematical Thinking (ACT) framework.

Classroom Culture

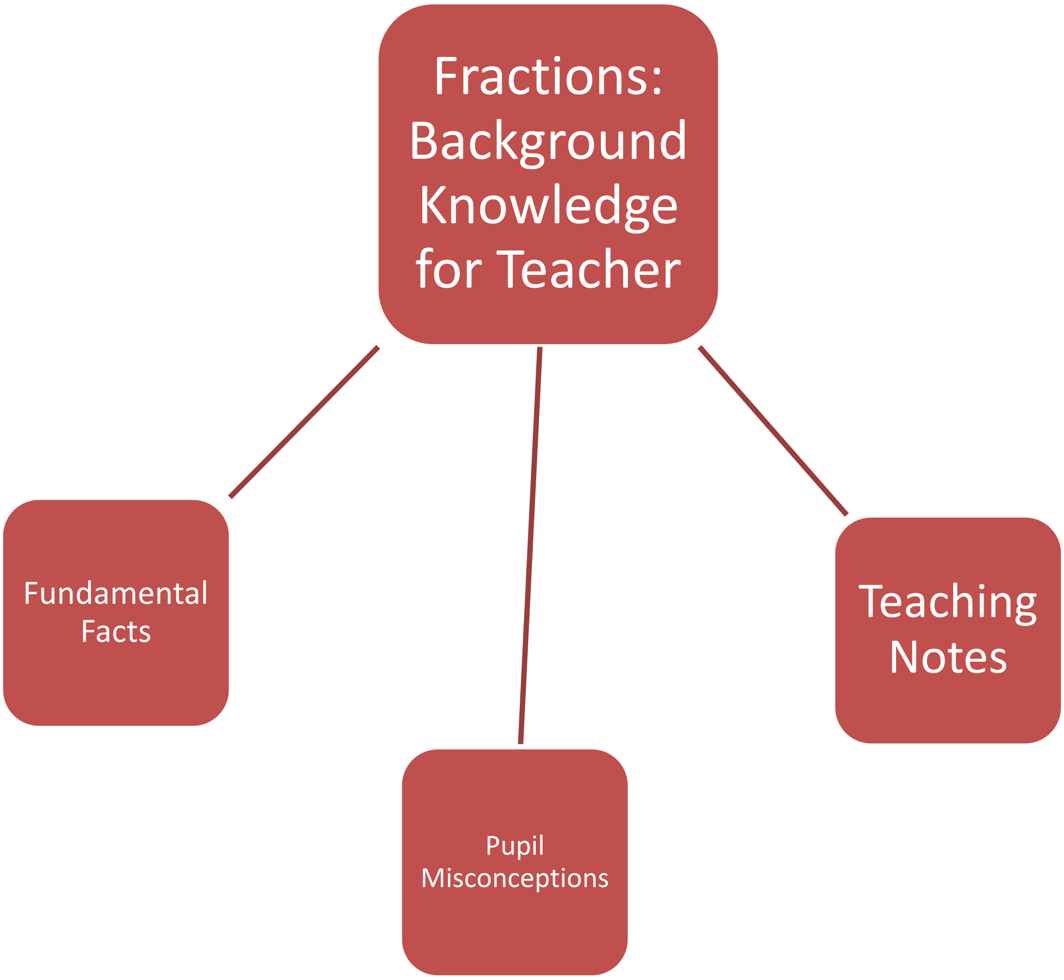
Creating and maintaining the correct classroom culture is a pre-requisite for developing and enhancing mathematical thinking. This requires the teacher to:

- cultivate a ‘have ago’ attitude where all contributions are valued;
- emphasise the importance of the process and experimenting with various methods;
- facilitate collaborative learning through whole-class, pair and group work;
- praise effort;
- encourage pupils to share their ideas and solutions with others;
- recognise that he/she is not the sole validator of knowledge in the mathematics lesson;
- ask probing questions (see Appendix B for a list of sample questions and sample teacher language);
- expect pupils to grapple with deep mathematical content;
- value understanding over ‘quick-fix’ answers; and
- use revoicing[1] (reformulation of ideas) as a tool for clarifying and extending thinking.

In this type of classroom pupils are expected to:

- share ideas and solutions but also be willing to listen to those of others; and
- take responsibility for their own understanding but also that of others.

[1] [1] Revoicing is ‘the reporting, repeating, expanding or reformulating a student's contribution so as to articulate presupposed information, emphasise particular aspects of the explanation, disambiguate terminology, align students with positions in an argument or attribute motivational states to students' (Forman & Larreamandy-Jones, 1998, p. 106).


Fractions: Background Knowledge for Teacher
Fundamental Facts
Pupil Misconceptions
Teaching Notes

FRACTIONS: BACKGROUND KNOWLEDGE FOR TEACHERS

Fundamental Facts about Fractions

1. Fractional parts are equal shares or equal-sized portions of a whole or unit (Van de Walle, 2007). There are two main ways when finding these types of numbers (numbers which are not whole numbers). Firstly, in measurement, the length, height, width, capacity, etc. of an object may fall between two whole numbers. Secondly, situations where quantities are shared often require numbers other than whole numbers.
2. Fractions can also represent quantities greater than one, that is $\frac{3}{2}$ $\frac{5}{4}$, etc.
3. Fractions represent a number but also a ratio
4. Fractions can be represented as:
    a. part of a whole;
    b. a place on the number line;
    c. an answer to a division calculation; or
    d. a way of comparing two sets or measures.
5. Fractions can chiefly be considered in three broad categories: Rational Fractions, Fractions as Operators and Equivalent Fractions[2].

    - *Rational Fractions* are simply a way of representing sizes that are not whole numbers, for example, if a pizza is cut into 4 equal parts and you ate 1 slice of the pizza, you didn't eat the whole pizza, you ate one slice of the four slices ($\frac{1}{4}$).
    - *Fractions as Operators* refer to instances where the fraction acts like an operator in that they tell us to do something with the whole number, for example, 30 sweets divided equally amongst 5 pupils - the fraction is telling us to do something with the 30 and the link with division is clear. The 30 needs to be divided by 5 giving each child 6 sweets. Taking a less simple example, in $\frac{3}{8}$ of 24 the fraction is telling us to divide 24 into 8 equal groups and then to highlight/select 3 of these groups. Thus, the denominator is the divisor and the numerator is a multiplier (indicating a multiple of the particular fractional part).

[2] Suggate, Davis & Goulding (2010)

- *Equivalent Fractions* are two (or more than two) ways of describing the same amount by using different-sized fractional parts. The ratio between different numbers can give different representations of the same fraction. They enable us to write the same amount in multiple ways, for example, $\frac{2}{4} = \frac{1}{2} = \frac{4}{8}$, etc. This concept of equivalent fractions is very important later when pupils have to add and subtract fractions so appropriate time and energy should be taken to present it in a meaningful way in the early stages.

6. All rational numbers (any number that can be expressed as a ratio of two whole numbers) have equivalent representation as fractions.
7. Fractions need to belong to the same 'family' in order to add or subtract them, that is, the denominator must be the same. In some instances, this requires adjusting the fractions so that they have a common denominator. It is important that this adjustment preserves the ratio between the numerator and the denominator. This adjustment to the same 'family' is not necessary when multiplying or dividing fractions.

8. It is usual to express a fraction in its lowest terms, for example, in $\frac{5}{20}$ both the numerator and the denominator are divisible by 5 so it can be written in its lowest terms as $\frac{1}{4}$.The lowest term means that there are no common factors in the numerator (top) or the denominator (bottom).

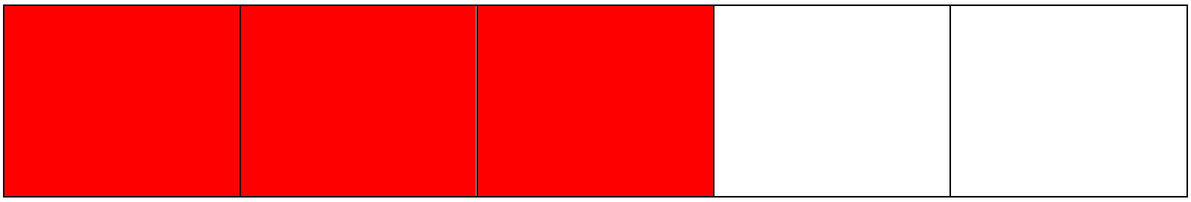
Possible Pupil Misconceptions involving Fractions

- Even when pupils grasp the basic concept of fractions they may still believe that $\frac{6}{14}$ is bigger than $\frac{4}{8}$ just because the numbers are bigger.

- Pupils often find fractions difficult to grasp because it is counter-intuitive considering their previous experience with whole numbers, for example, the larger a denominator then the smaller the fraction size. Whole number ideas can actually interfere with the development of fractions in the early stages.

- Similarly, having learned that whole numbers can be written in one way, it may be difficult for pupils to grasp that the same fraction can be written in a large variety of ways, for example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{25}{50}$, etc. Much practice and discussion is necessary to solidify the concept of equivalence.

- Social conventions can restrict the possible fractions within a situation[3], for example, pupils may assume that a visual diagram always represents the number 1.



Thus, a pupil may identify this fraction diagram (correctly) as representing $\frac{3}{5}$ or $\frac{2}{5}$ or both; However, they are less likely to see other possible representations, for example, $1\frac{2}{3}$, $2\frac{1}{2}$, $1\frac{1}{2}$ or $\frac{2}{3}$ These latter representations are made possible when it is understood that the whole unit can represent numbers other than the number 1.

- Pupils may be tempted to add fractions which have different denominators without subdividing them into parts (or families) which are the same size, for example, $\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$ because they just add the 'tops' and the 'bottoms'. Similarly, in subtraction pupils may be tempted to use the same procedure, for example, $\frac{5}{6} - \frac{1}{4} = \frac{4}{2}$.
- In multiplication, pupils may attempt to use the procedure which they learned for adding fractions, for example, $\frac{2}{3} \times \frac{1}{5} = \frac{3}{8}$.

Teaching Notes

Replacing rules and rote with understanding

A global understanding of the concept of fractions[4] and a deep sense of their purposes is more important than learning a set of rules. For this reason the introduction of rules should be delayed until pupils have arrived at a complete understanding of a concept. This demands that the methodologies of class discussion, problem solving and concrete experiences are central to instruction. In such learning environments, pupils are facilitated to develop their own understandings and it is from those understandings that conceptual and algorithmic understandings are reached. Otherwise the learning of fractions "is relegated to rules without reasons"[5]

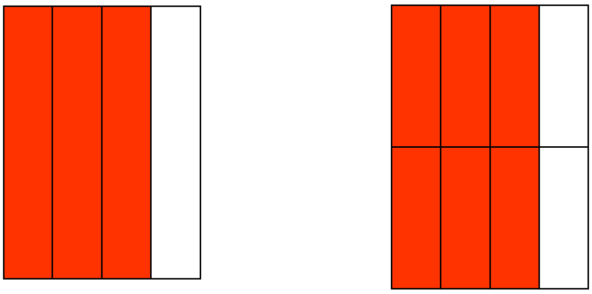
Sharing

[3] Askew (2000); Anghileri (2007).
[4] Anghileri (2007)
[5] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.286.

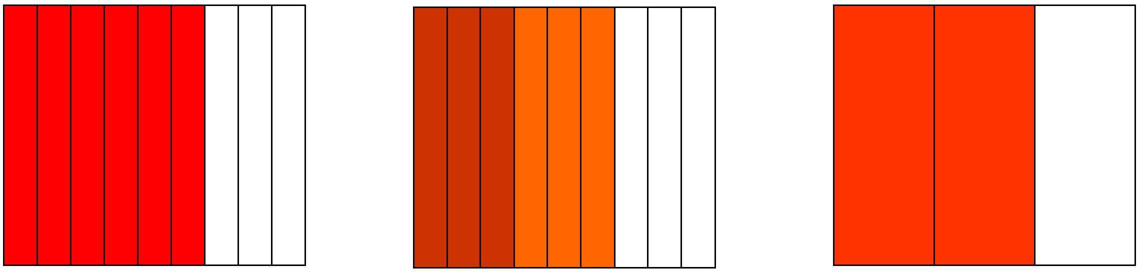
Focus on the concept of sharing can be helpful when introducing fractions as it meaningfully connects the idea of fair shares with fractional parts. Moreover, it assists the learner in understanding that the larger the denominator the smaller the fractional part, that is, the more sharers there are, the smaller the portions! It is a good idea to link this work with division.

Square or Circular Models

The habitual use of circular items such as pizzas and cake as models to teach fractions has certain shortcomings and a square unit can often be more versatile for teaching fractions[6]. For example, if a square is cut into vertical slices, some of which are shaded, it can also be further cut into horizontal slices without altering the amount shaded, for example, $\frac{3}{4}$ can be horizontally 'cut' to make $\frac{6}{8}$. This is not possible with a circular unit.



Furthermore, with a square unit it is also possible to use a 'sticker' which will combine strips to make new fractions[7]. For example, if a square is cut into 9 equal parts with 6 strips shaded, then if groups of 3 are stuck together it shows that $\frac{6}{9}$ is equivalent to $\frac{2}{3}$.



One-Dimensional Models

This concept can be taken a step further in that a one-dimensional linear representation of a fraction may be even more useful in developing understanding than using a circular or a square object. A line or a piece of string may be easier for pupils to understand because it can only be divided in one way

[6] Suggate et al (2010)
[7] Suggate et al (2010)

whereas two-dimensional objects can be divided in multiple ways. Therefore, length models may provide a stronger image for relative size than area models[8].

Estimation

Pupils must learn to estimate the size of an answer before completing an operation in order that they think deeply about the size of each fraction and predict what might happen because of the operation. Questions which might be helpful to scaffold estimation include:

- How big is the fraction?
- Which whole number is it nearest to?
- When you complete the operation will the answer be bigger or smaller than the fractions in the operation?
- How much bigger/smaller will the answer be?
- Estimate what that answer might be?

Multiple Representations

Ideally, pupils should have opportunities to represent fractions in multiple formats, for example, using concrete materials (including beads, counters, cubes, etc.); other pupils; graphical representations (including circular, rectangular, square, etc.) requires a deep understanding of fractions and can be a good stimulus for group and whole-class discussion. Similarly, these activities can sometimes be viewed as opportunities for informal assessment.

Evidence suggests that the use of **a variety of representational models**[9] is crucial in the teaching of fractions, not only to introduce concepts but as a means of clarifying ideas that may appear confusing to pupils in symbolic form. It is often valuable to repeat the same activity using different models. Three main types of models can be distinguished: area models, linear models and set models.

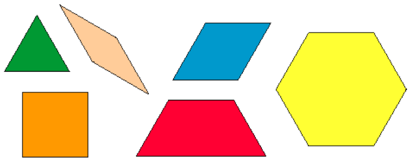
[8] Lewis, Perry, Friedkin & Baker (2010)
[9] Askew (2012); Boaler (2009); Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009)

Area Models

These models represent fractions as part of an area or a region and are useful when exploring fractional parts of 2D shapes. Typical examples of this type of model are:

- Circular "pie" pieces where a full circle represents a whole
- Rectangular pieces where any piece can represent a whole

- Shapes/Areas created with rubber bands on a geoboard
- Shapes/Areas drawn on dotted paper or grids
- Pattern blocks – multiple polygons
- Folded paper



The appearance of labels on manipulatives may appear helpful but they can also deprive pupils of opportunities to think about the size of the pieces relative to the whole. It also fuels the assumption that only one of the pieces can be considered as the whole.

Length Models

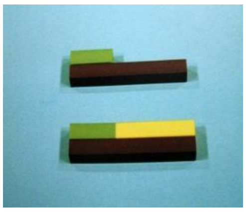
These are measurement models where lengths are compared and fall into two main categories:

1. *Linear*: Lines which are drawn and subdivided (number lines or line segments). The number line is considered to be particularly helpful due to the difference between

a) plotting an actual number on a number line and noting its distance from 0

b) simply comparing one length with another

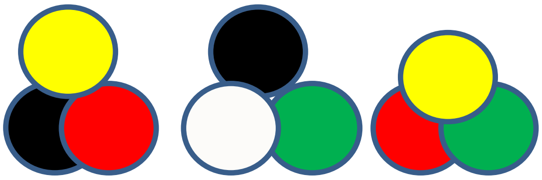
2. *2D and 3D:* Concrete materials that can be compared in terms of their length are folded paper strips, ribbon, string (2D) or fraction bars/rods (3D)



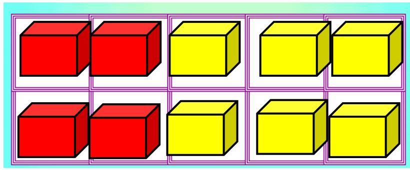
Set Models

In this case the whole is considered as a set or a collection of objects (for example, counters) with subsets making up the fractional parts. Typical examples include:

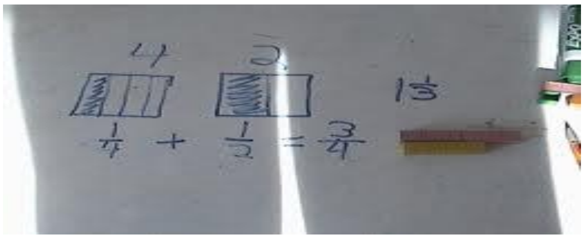
- Partitioned collections of objects



- Arrays- rows and columns of objects



- Drawn representations of objects in sets

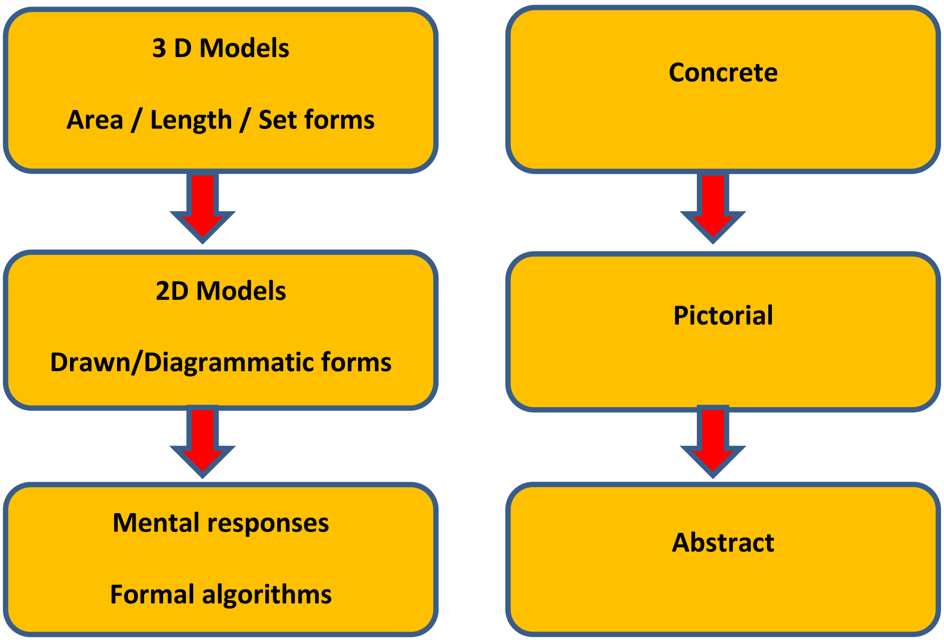


Wherever possible, the pictorial recording of each of the set models should be encouraged, followed by an oral description of the representation.

Regardless of the model used, regular use of the terms **whole, one whole** or **one** is important so that pupils are reminded to constantly make comparisons of fractional parts to the whole.

Teaching and learning fractions: From concrete to abstract

The teaching and learning of concepts along the trajectory follows a similar structure



Concrete/practical representations involving manipulatives and 3D materials often used to introduce a concept. The Area, Length and Set models of exploring fractions are applicable here.

Pictorial/Drawing/Diagrammatic representations are used to consolidate the concept and to provide an alternative visual model. The Area, Length and Set models can all be represented in this way.

Abstract representations are useful for encouraging mental responses and to develop the algorithm. As pupils gain confidence in concrete and pictorial activities they are enabled to offer mental responses before finally recording in algorithmic form.

Fractions Learning Trajectory

Fractions Learning Trajectory Level A[10]

Trajectory Levels	Concept	Developmental Experiences: Concrete	Pictorial	Abstract
	Level A.1 **Find $\frac{1}{2}$ of a set or shape**	**Linear** • fraction bars • folded paper strips • pieces of string • cuisenaire rods **Area** • pattern blocks • paper folding • geoboards • pegboards **Set** • pupils • collections of objects • rows and columns of objects	**Linear** • line segments • number lines • empty number lines **Area** • shape drawing • 10×10 grid drawing • dotted paper drawing **Set** • pupil representations of sets (drawing)	
	Level A.2 **Find $\frac{1}{4}$ of a set or shape**	***As for Learning Experiences in Level A.1***		
	Level A.3 **Given $\frac{1}{2}$ or a $\frac{1}{4}$ find a whole unit**	***As for Learning Experiences in Level A.1***		

[10] Generally aligned to the objectives for 1st and 2nd classes (see page 6)

Fractions Learning Trajectory Level B[11]

Trajectory Levels	Concept	Developmental Experiences: Concrete	Pictorial	Abstract
	Level B.1 **Compare, order, count and identify fractions and equivalent fractions with denominators 2,4,8,10**	**Linear** • fraction bars • folded paper strips • string and paper clips • cuisenaire rods **Area** • pattern blocks • paper folding • geoboards • pie pieces **Set** • collections of objects	**Linear** • fraction walls • parallel number lines • empty number lines • benchmarks number line **Area** • shape drawing • 10×10 grid drawing • benchmarks pictures **Set** • pupil representations of sets (drawing)	• symbolic is recorded with pictorial • benchmarks (0, ½, 1) • revisit relevant table facts • exchange 4/4 for 1 whole when counting orally, eight quarters for 2 wholes, etc. • compare fractional pairs only using benchmarks concepts
	Level B.2 **Calculate a unit fraction (with denominators 2,4,8,10) of a whole number** e.g. find $\frac{1}{4}$ of 12	**Set** • collections of objects	**Set** • partitioning of objects	• symbolic is recorded with pictorial • link with division facts
	Level B.3 **Calculate multiple fractions (with denominators 2,4,8,10) of a whole number** e.g. find $\frac{3}{4}$ of 12	**Set** • collections of objects	**Set** • partitioning of objects	• symbolic is recorded with pictorial • link with multiplication and division facts
	Level B.4 **Calculate the number given the unit fraction (with denominators 2,4,8,10)** e.g. $\frac{1}{8}$ of a number is 6. What is the number?	**Set** • collections of objects	**Set** • partitioning of objects	• symbolic is recorded with pictorial • link with multiplication and division facts

[11] Generally aligned to objectives for 3rd and 4th classes (see page 7).

Concept	Developmental Experiences: Concrete	Pictorial	Abstract
Level B.5 **Calculate the number given the multiple fraction (with denominators 2,4,8,10)** e.g. $\frac{3}{8}$ of a number is 9 what is the number	**Linear** • unifix towers • folded paper strips • cuisenaire rods **Area** • pattern blocks • pegboards • geoboards **Set** • collections of objects • rows and columns	**Linear** • line segment • empty number lines **Area** • shape drawing • 10×10 grid drawing • dotted paper drawing **Set** • pupil representations of drawings	• symbolic is recorded with pictorial • identify patterns • conjecture/proof
Level B.6 **Compare, order, count and identify fractions and equivalent fractions with denominators 3,5,6,9,12**	***As for Learning Experiences in Level B.1***		
Level B.7 **Calculate a unit fraction (with denominators 3,5,6,9,12) of a whole number** e.g. what is $\frac{1}{5}$ of 10?	***As for Learning Experiences in Level B.2***		
Level B.8 **Calculate multiple fractions (with denominators 3,5,6,9,12) of a whole number** e.g. what is $\frac{3}{5}$ of 20?	***As for Learning Experiences in Level B.3***		

	Concept	Developmental Experiences Concrete	 Pictorial	 Abstract
	Level B.9 **Calculate the number given the unit fraction (with denominators 3,5,6,9,12)** e.g. $\frac{1}{6}$ of a number is 18. What is the number?		***As for Learning Experiences in Level B.4***	
	Level B.10 **Calculate the number given the multiple fraction (with denominators 3,5,6,9,12)** e.g. $\frac{3}{6}$ of a number is 24. What is the number?		***As for Learning Experiences in Level B.5***	

Fractions Learning Trajectory Level C[12]

Trajectory Levels	Concept	Developmental Experiences: Concrete	Pictorial	Abstract
	Level C.1 **Identify, construct, compare, order and count improper fractions**	***As for Learning Experiences in Level B.1***		
	Level C.2 **Express improper fractions as mixed numbers and vice versa** e.g. $\frac{6}{5}$ is the same as $1\frac{1}{5}$	**Linear** • folded paper strips • string and paper clips • cuisenaire rods **Area** • pattern blocks • pie pieces **Set** • collections of objects	**Linear** • fraction walls • empty number lines • benchmarks number line **Area** • shape drawing • 10×10 grid drawing • benchmarks pictures **Set** • pupil representations of sets (drawing)	• symbolic is recorded with pictorial • benchmarks (1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, etc.)
	Level C.3 **Construct algorithm for equivalent and simplified fractions**	**Area** • selection of paper strips	**Area** • shape drawings	• revisit relevant table facts • algorithm

[12] Generally aligned to the objectives in 5th and 6th classes (see page 8)

Fractions Learning Trajectory Level D[13]

Trajectory Levels	Concept	Developmental Experiences: Concrete	Pictorial	Abstract
	Level D.1 **Add and subtract fractions (first with like denominators, then repeat with unlike denominators)**	**Linear** • fraction bars • cuisenaire rods **Area** • pie pieces • pegboards **Set** • collections of objects	**Linear** • line segments • empty number lines **Area** • shape drawing **Set** • pupil representations of sets	• symbolic is recorded with pictorial • identify patterns • conjecture • estimation
	Level D.2 **Add and subtract mixed numbers** e.g. $1\frac{1}{3} + 3\frac{2}{6}$	*As for Learning Experiences in Level D.1*		
	Level D.3 **Multiply a fraction by a whole number** e.g. $\frac{2}{3} \times 6$	**Linear** • folded paper strips **Area** • pie pieces • pattern blocks **Set** • collections of objects	**Linear** • empty number lines (using benchmarks to informally deduce answer using equivalence) **Area** • shape drawing **Set** • pupil representations of sets	• estimation using benchmarks • symbolic is recorded with pictorial • repeated practice should lead to a recognition of pattern and a deeper understanding of the algorithm when introduced • commutative rule e.g. ¾ of 6 = 6 lots of ¾
	Level D.4 **Multiply a fraction by a fraction** e.g. $\frac{1}{4} \times \frac{2}{5}$	*As for Learning Experiences in Level D.3*		

13 Generally aligned to objectives in 5^{th} and 6^{th} classes (see page 8)

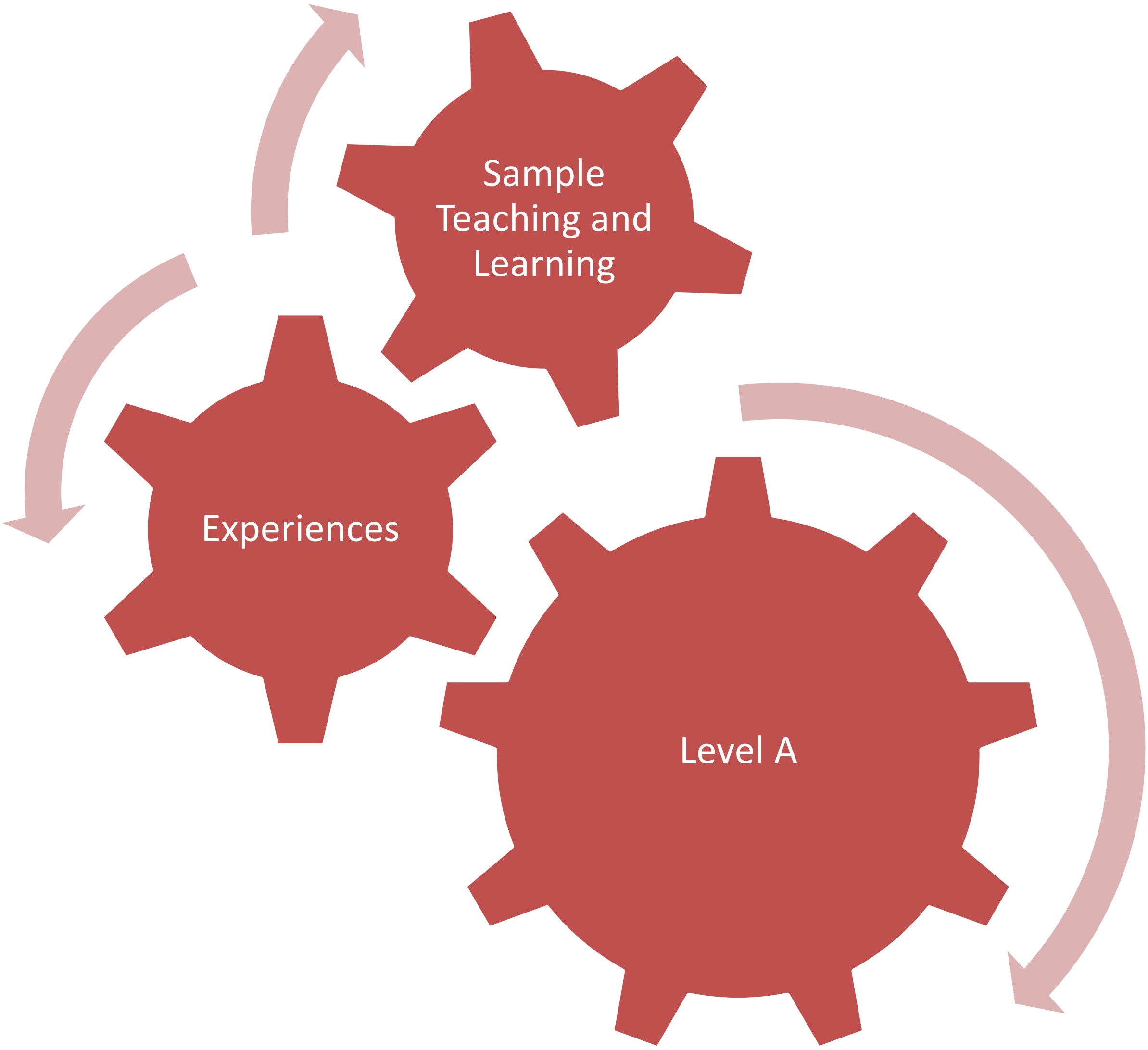
	Concept	Developmental Experiences		
		Concrete	Pictorial	Abstract
	Level D.5 **Divide a whole number by a unit fraction** e.g. $2 \div \frac{1}{4}$	**Linear** • string and paper clips • fraction bars • cuisenaire rods • folded paper strips **Area** • pattern blocks	**Linear** • empty number lines **Area** • shape drawing	• estimation using benchmarks • symbolic is recorded with pictorial • repeated practice should lead to a recognition of pattern and a deeper understanding of the algorithm when introduced

Fractions Learning Trajectory Level E[14]

Trajectory Levels	Concept	Developmental Experiences: Concrete	Pictorial	Abstract
	Level E.1 **Understand and use simple ratios**	**Linear** • ribbons • unifix cubes • cuisenaire rods **Set** • collections of objects • rows and columns of objects	**Set** • pupil representations	• symbolic is recorded with concrete and pictorial

At all stages, pupils need opportunities to talk about their thinking and to articulate their ideas. This not only clarifies and consolidates concepts but allows the teacher to assess the level of understanding. If a child is showing difficulty at any stage, they would benefit from additional experiences from the previous activities to support their learning needs.

[14] This objective relates to 6th class (see page 8)


Sample
Teaching and
Learning
Experiences
Level A

Fraction Trajectory Level A

Sample Teaching and Learning Experiences

Level A.1	Level A.2	Level A.3
find half of a set or shape	find half of a set or shape	given half or quarter find a whole unit

Preliminary Work

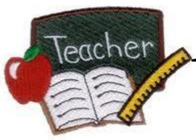
As already stated, the first step in developing fractions should help pupils construct the idea of fractional parts of a whole – that is when the whole has been partitioned the resulting parts are equal sized or fair shares. Because pupils seem to understand the idea of fair sharing, this can be a good starting point for introducing fractions. Although some of the examples below may appear sophisticated for the early stages of teaching fractions, these are NOT intended to form part of the explicit and formal instruction. Rather they are a means of allowing pupils to explore real life problems associated with sharing and can provide the teacher with an initial idea of how to formally introduce fractions

Informal Sharing

1. Consider simple problems:

- Share 6 brownies equally between 2 pupils
- Share 8 brownies equally between 4 pupils
- Share 10 brownies equally between 5 pupils

Pupils will typically share the items by distributing one of them at a time. The teacher should allow the pupils to proceed with what naturally occurs to them but deliberately prompt the pupils with questions such as:



Are the brownies equally shared?
How do you know?
How will we check?
How many brownies did each get?

Remember

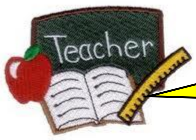
We are not yet exploring fractions here in terms of fractions of a whole number, that is, halves, quarters, etc. We are more concerned with the concept of **equal sized portions.**

2. When problems involving left over pieces are presented, pupils have to think of subdividing some of the items and are forced to challenge their existing understanding of equal sharing.

Consider problems which involve more items than sharers:

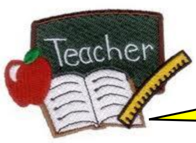
- Share 5 brownies equally between 2 pupils
- Share 10 brownies equally between 4 pupils
- Share 6 brownies equally between 4 pupils

The pupils may proceed to distribute one at a time until they discover that there are not enough whole leftovers! Teachers should seize this opportunity to problem solve with regard to the leftover pieces:



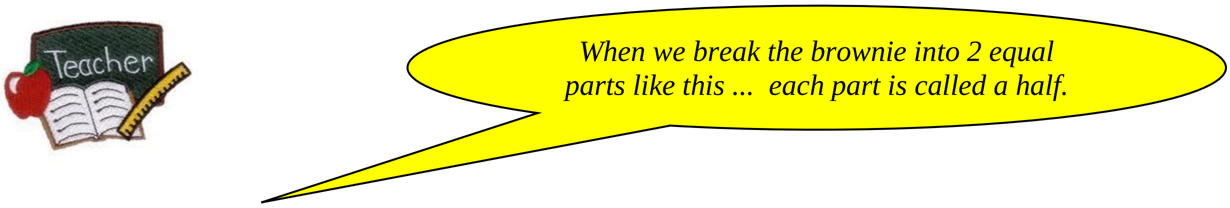
Explain how you shared the brownies.
Why have you stopped at these 2?
Can you explain the problem?
Does anybody have any ideas about how we could share these 2 leftover brownies?

The pupils may suggest dividing the leftover brownies in half and begin distributing the halves one by one. Teacher needs to promote more discussion here:



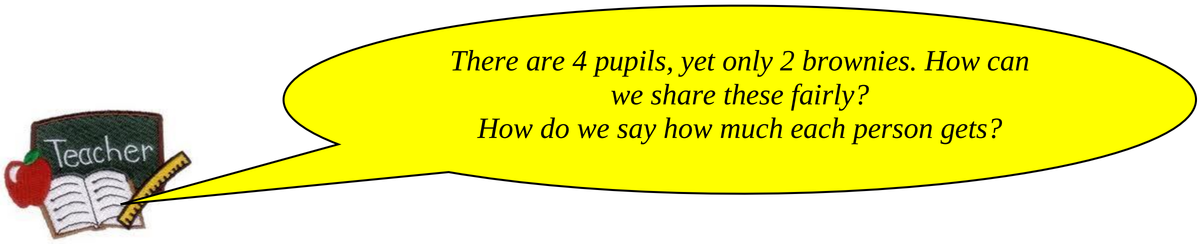
Are the brownies equally shared?
How will we check?
What could we call these smaller parts?
How do we say how much each person got?

The term 'half' can be casually introduced here. However, it should not involve any fraction symbolism. The emphasis at this point is that the number of parts that make up the whole (that is, the 2 parts that make up the whole brownie) gives us the name of the part or the share:



3. Consider problems which involve more sharers than items:
   - Share 2 brownies equally between 4
   - Share 4 brownies equally between 8

These examples further challenge the pupils as they will be forced to confront the idea of subdividing before distributing!

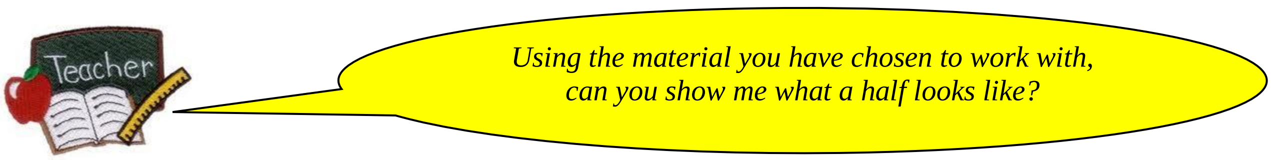


Level 1.1: Find $\frac{1}{2}$ of a set or shape

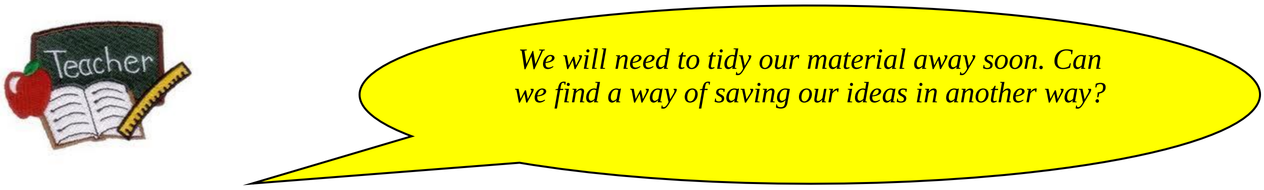
Linear Model

As already suggested, it can be helpful to begin with the linear model as the options for halving are more limited, therefore easier to do. The following steps may be followed:

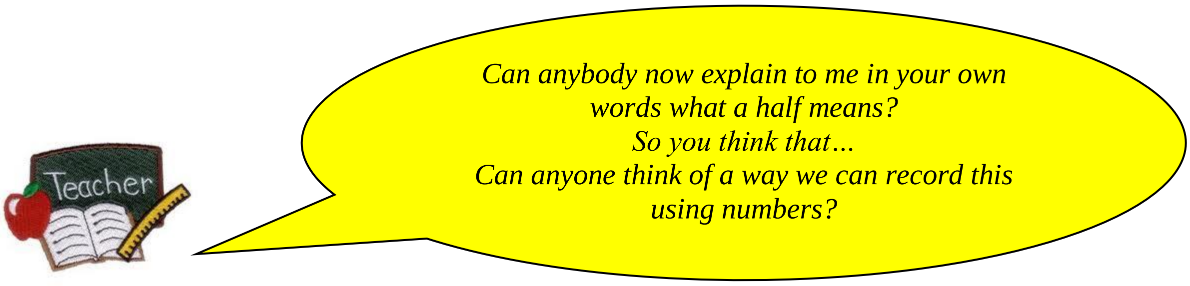
1. Teacher distributes a range of concrete materials which are suitable for the linear model, for example, fraction bars, Cuisenaire rods, string/paper clips, folded paper strips. Pupils are asked to work with the material of their choice.
2. Pupils are invited to show what a half looks like, for example:



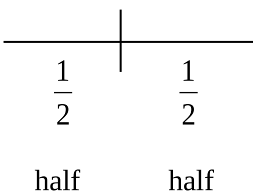
3. Time is allowed for sharing the various ideas/representations including the inaccurate ones in order to generate discussion.
4. Pupils are now invited to suggest ways of recording the representations in 2D form:



5. Depending on the material used the pupils might:
   - Draw/ Trace around the fraction bars/rods
   - Draw an empty line segment and indicate half way mark using an interval line
   - Paste a folded paper strip into their journal
   - Take a photograph of the 2D representation
6. Displays of concrete and pictorial representations will be useful reference points for pupils, in addition to being a record of pupils's work.
7. Using these representations, this is an opportune time for the teacher to introduce formal symbolism $\frac{1}{2}$:



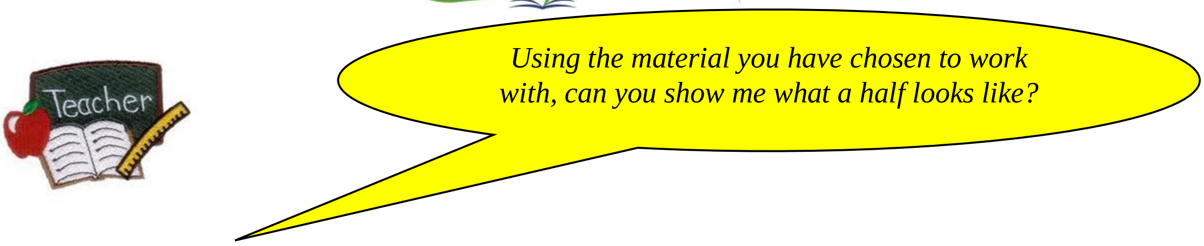
8. Teacher should add formal symbol of a half to the displays.



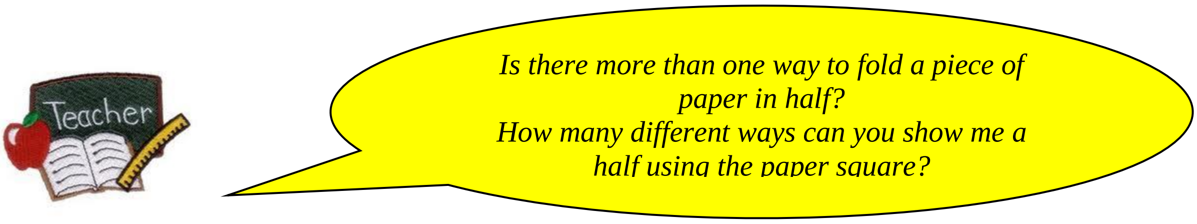
Area Model

Unlike linear models, area models are more open-ended and could be the next logical step during an exploration of halves.

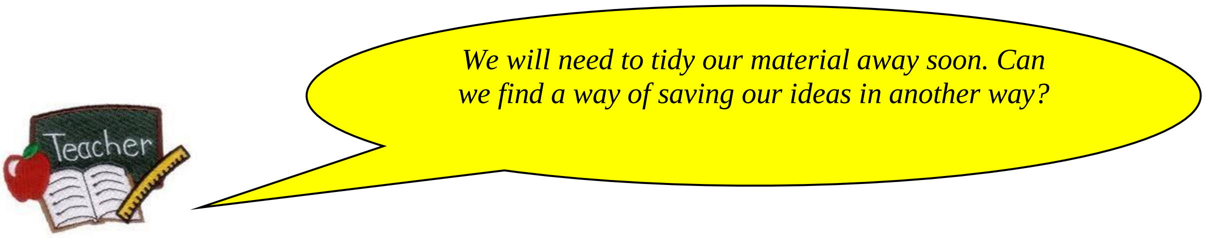
1. Teacher distributes a range of concrete materials which are suitable for the area model, for example, pattern blocks, geoboards, paper shapes of regular polygons (squares, rectangles, hexagons, triangles) Pupils are asked to work with the material of their choice.
2. Pupils are invited to show what a half looks like, for example:



3. Time is allowed for sharing the various ideas/representations including the inaccurate ones in order to generate discussion. Because the area model provides multiple solutions, keep the activity as open-ended as possible!



4. Pupils are now invited to suggest ways of recording the representations in 2D form:



5. Depending on the material used the pupils might:
   - Draw/Trace around the pattern blocks/2D shapes, shading half of the shape
   - Draw an outline of a shape on a 10 by 10 grid and colour in half of the squares
   - Use dotted paper to record shapes made on the geoboard shading in to show half
   - Take a photograph of the 2D representation
6. Using these representations, this is an opportune time for the teacher to reinforce the formal symbolism $\frac{1}{2}$.
7. If the teacher has explored the linear model with the pupils, it may be useful to add the above representations of the area model to those of the linear model.
8. The following activity may be helpful to consolidate pupils' understanding of halves and quarters in the context of the area model:

Fraction Game (for 2 players)[15]

Materials:

- Large laminated squares (32cm x 32cm as shown) – one per player
- Four smaller laminated shapes(8cm x 8cm) in 2 colours (red, blue) cut as shown
- A six sided die with customised faces as follows: $\frac{1}{2}\ \frac{1}{4}\ \frac{3}{4}\ \frac{1}{2}\ \frac{1}{4}\ \frac{3}{4}$

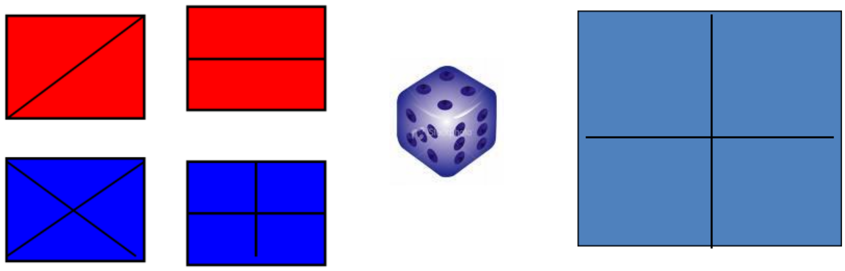
Preparation for this game may require a revision of open ended tasks such as:

Show me $\frac{3}{4}$ in as many ways as possible

3 square quarters using the blue square pieces

3 triangular quarters using the blue triangular pieces

$\frac{1}{2}$ and $\frac{1}{4}$ using one of the red pieces and a blue piece



The game is played in pairs where pupils take turns to throw the dice and put the appropriate piece onto the large square. The aim is to fill your 4 squares completely before your opponent. The game can also be played in reverse where pupils remove the pieces from the complete square until their square is empty. (See Appendix H for templates and a detailed explanation of this game)

Set Model

In set models the whole relates to a set of objects and the subsets are the fractional parts- a set of counters is a single entity which can make this difficult for pupils.

Common Misconception

Pupils often focus on the size of the set rather than the number of equal sets in the whole. For example, if 12 counters make the whole set, then a set of 3 counters is a quarter rather than the set of 4 counters. Four sets of 3 are needed to make the whole.

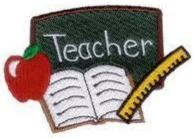
---

[15] Deboys and Pitt (1980)

1. The set model is best introduced using a root problem solved by the whole class:

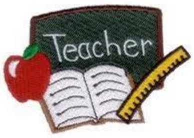
Share a set of 6 counters equally between 2 pupils

2. Pairs of pupils are given a set of six counters and asked:



Share the counters equally between you and your partner. How many did you get each?

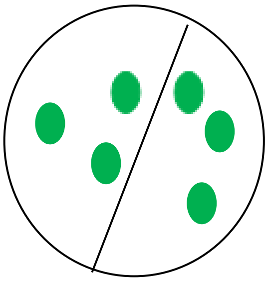
3. Using an area model folded in half, for example, a paper plate; ask the pupils to place their share in one of the halves. The language of $\frac{1}{2}$ is revisited as follows:



How many counters on the whole plate?
How many counters on half of the plate?

4. Pupils record the findings as drawings and as number stories:

$3+3=6 \quad \frac{1}{2}$ of $6 = 3 \quad 3$ is half of 6



5. Ask the pupils to draw the number story in another way and allow various representations to be shared with the whole class.
6. If the teacher has explored the linear model and/or area models with the pupils, teachers should add the examples of the set model representations to the existing linear and area models.

This creates a **living chart** representing fractions in multiple ways, that is, concrete, pictorial and abstract representations of the area, length and set models.

7. Repeat this problem using other number choices, for example: (8,2) (10,2) (12,2) (14,2) (16,2) (18,2)
8. Explore and discuss patterns, for example, doubles number facts. This is an ideal opportunity for revising.

Level 1.2: Find a $\frac{1}{4}$ of a set or shape

The methods described above can all be replicated to introduce quarters. A teacher may choose however to approach this in different ways depending on the learning needs of the pupils:

a) explore quarters after fully exploring halves using all 3 models (linear, area, set);
b) follow the exploration of halves for each model with a similar exploration of quarters; or
c) explore quarters concurrently with halves.

> Although the concept of equivalent fractions is further along the trajectory, teachers should not lose the opportunity to explore early understandings of the concept having fully explored halves and quarters.

1 whole			
$\frac{1}{2}$		$\frac{1}{2}$	
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Consolidation Activities for $\frac{1}{2}$ and $\frac{1}{4}$

The following activities may be helpful to consolidate pupils' understanding of halves and quarters in the context of the area model:

1. Dice Game (for 2 players)

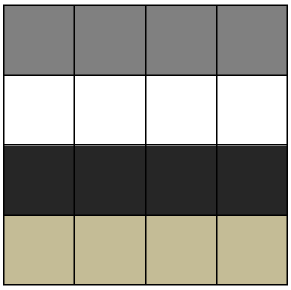
The dice game for fractions (see page 33) for halves can be used here for quarters.

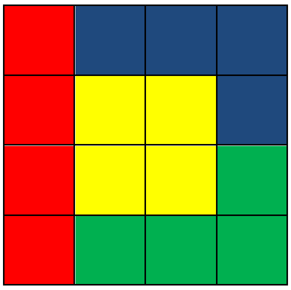
> **Common Misconception**
> When exploring area models, pupils often assume that the fractional parts must be the same shape as well as the same size.

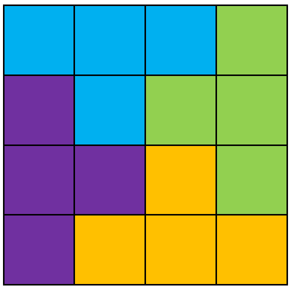
2. Conservation of Area

Activity 1:

The following open-ended activity challenges this misconception where pupils are asked to divide 4 × 4 grids into quarters in as many ways as possible:

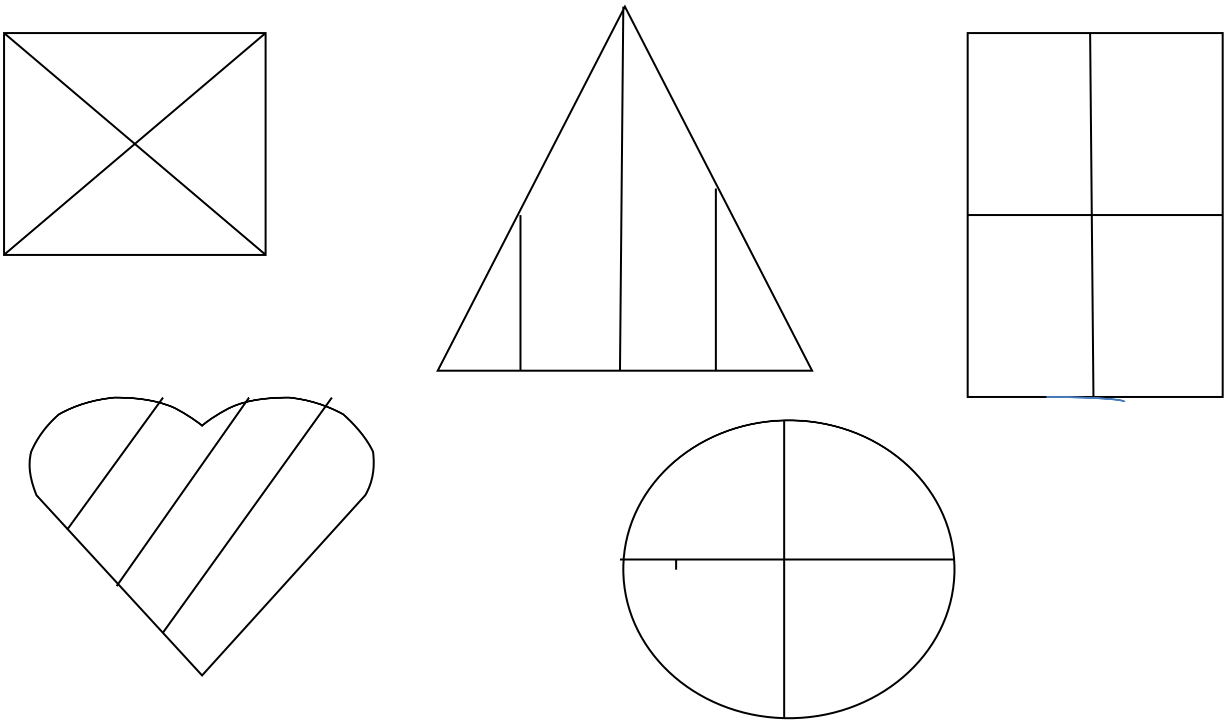


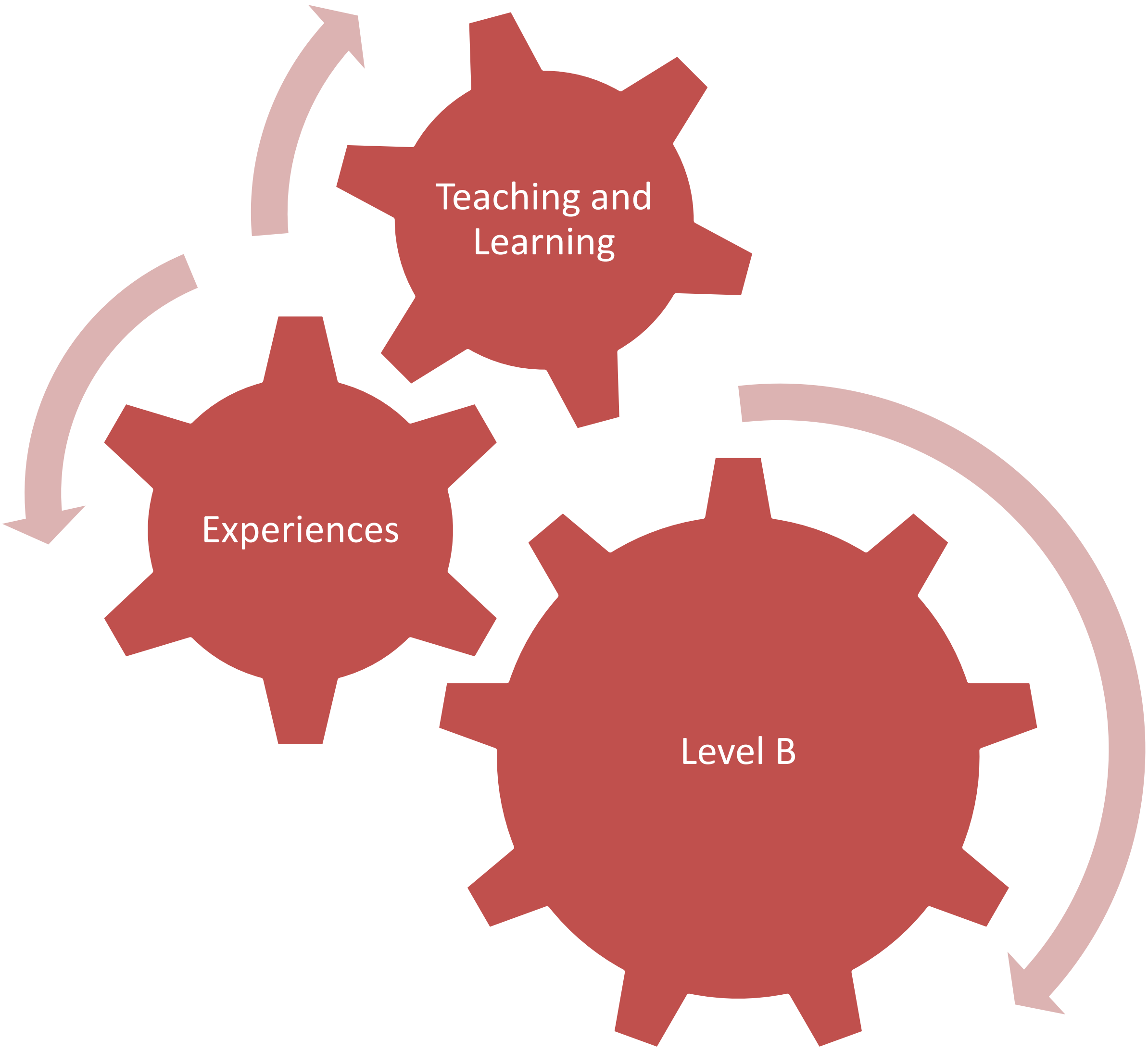




Activity 2: Find the Correct Shares

Pupils' mathematical ideas can be further challenged by presenting them with wholes that are correctly divided into requested fractional parts, that is, a quarter and those that are not. Pupils should be asked which of these figures are correctly divided into quarters and explain their reasons. The most important elements of this discussion are the incorrect examples.




Teaching and Learning
Experiences
Level B

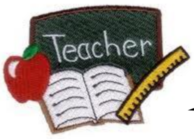
Identifying, Ordering and Comparing Fractions (including equivalent fractions)

1. Make a Fraction

Linear Model:

Paper Strips

In pairs, pupils use paper strips of various lengths to represent fractions.

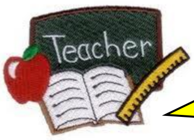


Show me eighths on 2 different lengths of paper.
Explain how you did it.
Is there any other way to do it?
Did someone do it in a different way to Mary?
Compare your eighths to your partner's eighths.

Give pupils 4 equal length paper strips. Ask them to represent halves on one strip, quarters on another strip, eighths on another strip, and tenths on the final strip.

Order your fraction paper strips from smallest to largest.
Which is the smallest?
Which is the largest?
Explain why this is.
John, can you explain what Mary said in your own words?

An extension of these activities is to ask pupils to identify multiple fractions. This can be done in a variety of ways including placing counters on fractional parts or shading fractional parts. This provides an ideal opportunity to introduce the mathematical terms numerator and denominator. Teacher modelling is critical here.



How many equal parts have you altogether in your whole paper strip?
So there are 8 equal parts. We call this number the denominator.
How many parts have you placed counters on?
So there are 3 parts out of 8 that have counters on them.
We call 3 the numerator.

Every opportunity should be taken to discuss the concept of numerator and denominator in a variety of settings.

String

The above activities can also be completed using lengths of string and paper clips. Here the focus could be on estimation (pupils estimate where to mark the fraction rather than folding).

Cuisenaire Rods

Each group of pupils have a box of Cuisenaire rods that they use to solve problems like the following:

- What fraction of the brown rod is the red rod?
- If the purple rod is $\frac{2}{3}$ which rod is the whole?

Pupils can also design their own questions.[16]

ICT Opportunities
Link: Interactive Cuisenaire rods

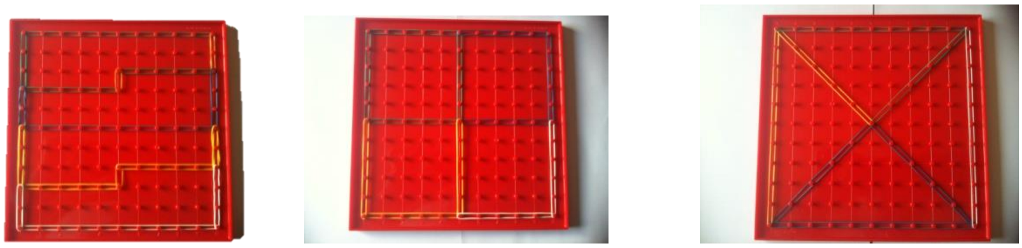
[16] Clarke, Roche and Mitchell (2008)

Area Model:

In addition to linear models, pupils need experience of identifying, comparing and ordering fractions in various area models.

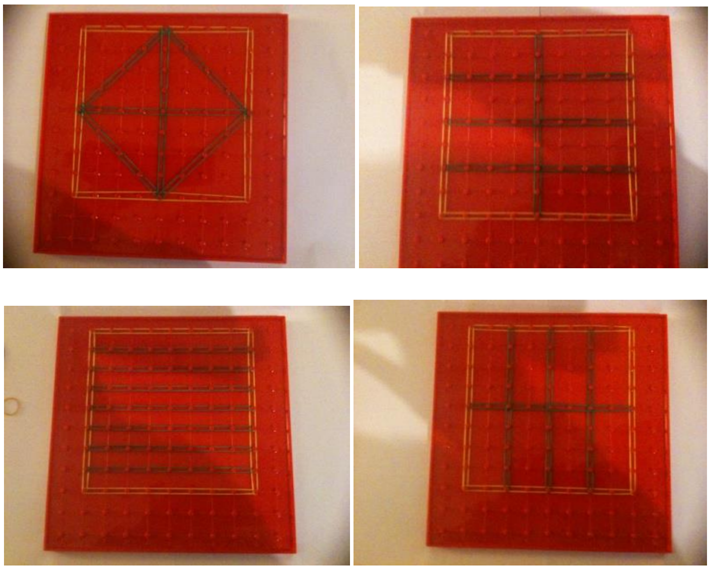
Geoboards

Show me quarters of the whole geoboard.
How many other ways can you divide your geoboard into quarters?
Is there another way of doing it?
Mary has an interesting idea which I think would be useful for us to hear.



These geoboards illustrate equal-sized quarters of a unit.

Pupils can also find fractions of an enclosed region of the geoboard.



ICT Opportunities
Link: Interactive Geoboard

These geoboards illustrate eighths in various forms. Note how different shapes can represent the same fraction. This provides rich opportunities for pupils to discuss and compare their findings.

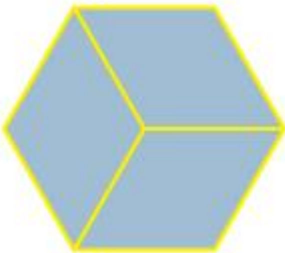
We will need to tidy our material away soon. Can we find a way of saving our ideas in another way?
John suggests that…
Could we also use dotted paper?

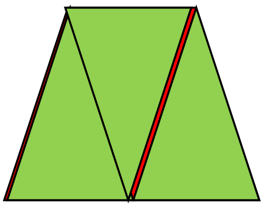
Dotted Paper

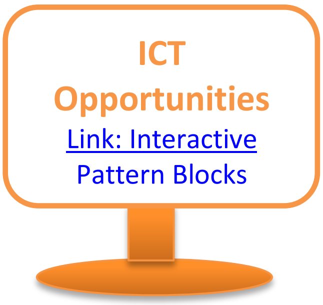
Similar activities to those completed with geoboards can be completed on dotted paper. Dotted paper can also be used to record the results of investigations using geoboards.

Pattern Blocks

Pattern blocks can be used for identifying and comparing fractions and equivalent fractions.



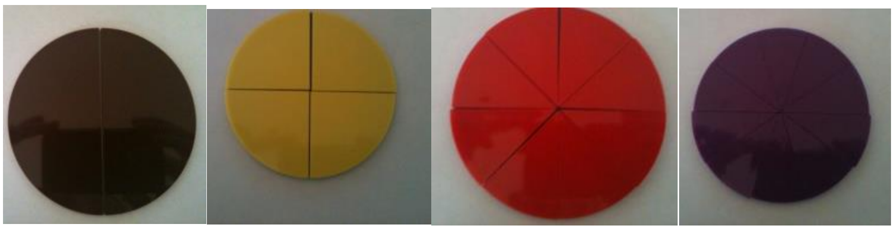




These pattern blocks illustrate thirds of the whole.

Pie Pieces

Pie pieces can be used for identifying, comparing and ordering fractions and equivalent fractions.



These pie pieces illustrate halves, quarters, eighths and tenths.

Rectangular Pieces

Rectangular pieces can be used for identifying, comparing and ordering fractions and equivalent fractions.

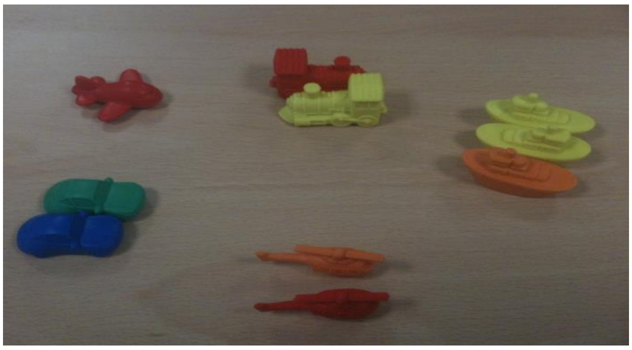
ICT Opportunities

[Link: Interactive fraction pieces]

These rectangular fraction pieces illustrate halves, quarters, eighths and tenths.

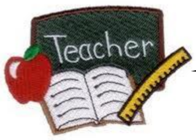
Set Model:

Extending the work done in Early Mathematical Activities (sorting, classifying, partitioning) sets of 10, 8 and 4 can be sorted for various attributes to identify fractional parts. In the example below a set of transport vehicles are used and can be sorted in various ways including colour, wheels, type of vehicle, flies/doesn't fly, etc.



Other sorting materials can also be used here, for example, attribute blocks, random collections, beads, buttons[17].

Elicit from the pupils what fraction of the whole set each subset makes, for example, 2 helicopters out of 10 vehicles, 3 ships out of 10 vehicles, 2 cars out of 10 vehicles, 1 aeroplane out of 10 vehicles, 2 trains out of 10 vehicles.



We will need to tidy our material away soon. How many different ways can we record what we discovered?
Choose a way of recording this.

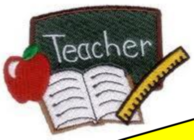
Some pupils might:

- draw a picture
- write about it
- take a photo

[17] Refer to Ready, Set, Go Maths manual for ideas regarding sorting materials.

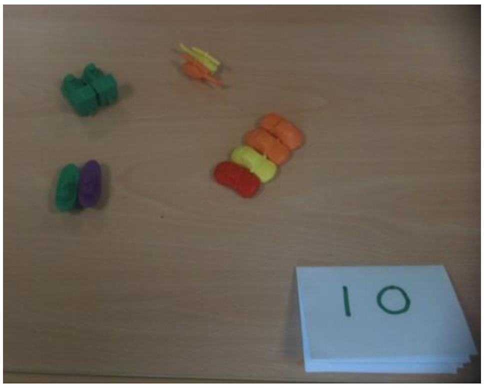
The concept of numerator and denominator should be introduced at this point. One way of doing this is for the teacher to model a way of recording the findings from the above activity, for example:

<u>2 helicopters</u>
10 vehicles



We have a set of vehicles. How many vehicles have we in our whole set?
The denominator is the number of parts in the whole set.
Here we have 10 in the whole set. 10 is the denominator.
How many helicopters have we in our whole set?
In this example the numerator is the number of helicopters out of our whole set of vehicles. We have 2 helicopters.

The sorting activity can be repeated to develop abstract representation. At this stage, a digit card can be used as a reminder that the unit set is 10 hence the fractions will be tenths.



Pupils are now encouraged to represent their findings as abstract fractions, for example

<u>2 ~~helicopters~~</u>
10 ~~vehicles~~

becomes $\frac{2}{10}$

ICT Opportunities
Link: Comparing fractions

Comparing Fractions

Benchmarks

Benchmarks are helpful for pupils to develop their number sense in relation to fractions. Estimation is central to applying benchmarks. Pupils need lots of opportunities to compare fractions to benchmarks. The most important reference points or benchmarks for fractions are 0, $\frac{1}{2}$ or 1. The following activities can be used to develop and consolidate benchmarking.

1. More less or equal to one whole [18]

Distribute collections of unit fractional parts to pupils. Each collection should contain multiple pieces of the same fraction. A combination of linear, area and set models could be used. Indicate to the pupils what kind of fractional part they have, for example, 'This bag contains eighths. This bag contains quarters.' Pupils must decide if their collection is less than one whole, equal to one whole or more than one whole. Pupils should discuss and record their findings using pictures or symbols.

2. **Fraction Pairs Task**[19]

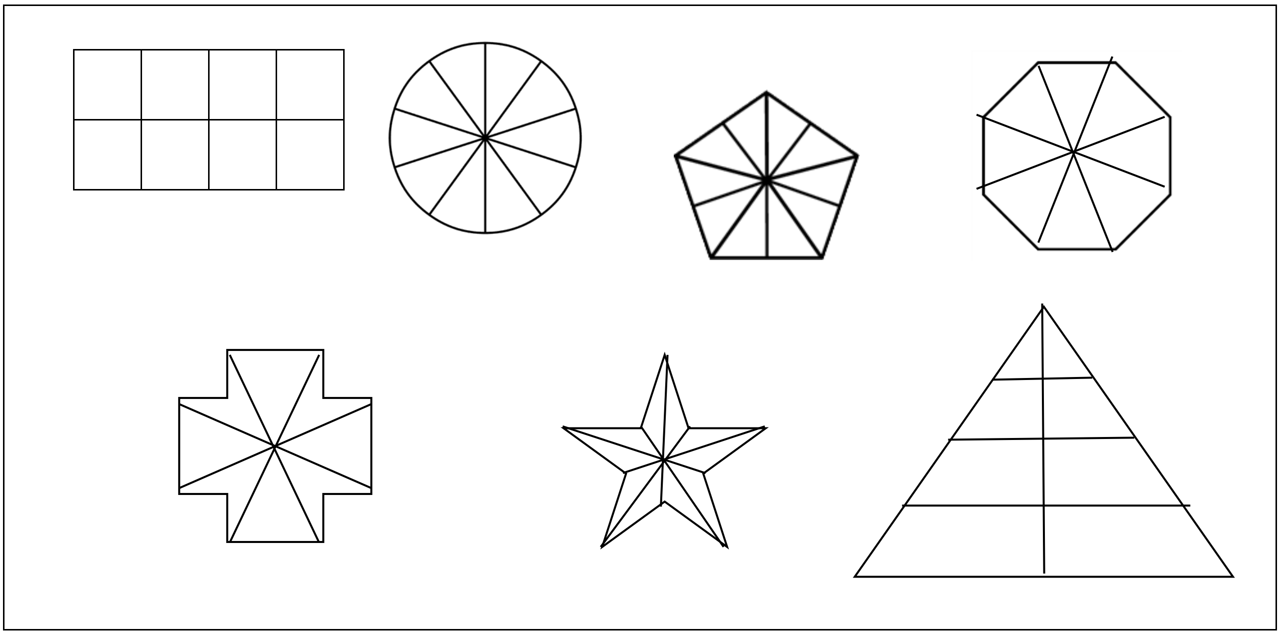
Provide pupils with eight pairs of fractions and ask them to decide, for each pair, which fraction is larger. It is important that pupils give you a rationale for this. The comparison should be made in their heads.

Fraction Pairs	Which Fraction is larger? Record your strategy.
a. $\frac{3}{8}$ $\frac{7}{8}$	
b. $\frac{1}{2}$ $\frac{5}{8}$	
c. $\frac{3}{4}$ $\frac{4}{5}$	
d. $\frac{2}{4}$ $\frac{4}{8}$	
e. $\frac{2}{4}$ $\frac{4}{2}$	
f. $\frac{3}{8}$ $\frac{5}{6}$	
g. $\frac{5}{6}$ $\frac{7}{8}$	
h. $\frac{3}{4}$ $\frac{7}{9}$	

[18] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p. 295
[19] Clarke, Roche and Mitchell (2008)

3. Find the Correct Shares activity

The fraction identification activity found at the end of Level A (activity 2 page 38) can be extended to include eighths and tenths.



Consolidation Activities

Dice Game

The dice game in Level A can also be extended to include eighths and tenths. (See Appendix H for templates and a detailed explanation of this game)

Fish for Fractions

This game is available in Appendix C.

ICT Opportunities
Fraction Track

Counting in Fractions

Teaching fractions should involve a lot of oral work and counting (see appendix D).

Level B.2

Calculate a unit fraction (with denominators 2, 4, 8, 10) of a whole number

1. The set model is best introduced using a root problem solved by the whole class:

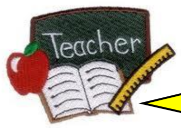
Share a set of 20 counters equally between 4 pupils

Or What is $\frac{1}{4}$ *of 20 counters?*

2. Ask pupils to estimate how many counters each pupil will get and to explain their reasoning.
3. Pairs of pupils are given a set of 20 counters and asked:

How can you show this using the counters?
Could someone else tell us a different way of doing it?

4. Pupils record the findings as drawings and as number stories.
5. Allow various representations to be shared with the whole class.
6. Repeat this problem using eighths and tenths, for example: (24,8) (30,10) (32,8) (40,10)
7. Explore and discuss patterns, for example, division facts. This is an ideal opportunity for identifying the algorithm which in this example is 20 ÷ 4 = 5.

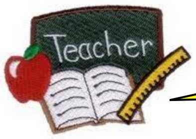


How could we do this in a quicker way?
Check if this works for other examples.
How can we record the findings using symbols?
Is this the most efficient way of doing it?
Which way is easiest to understand and why?

Level B.3
Calculate multiple fractions (with denominators 2, 4, 8, 10) of a whole number

Follow the steps as outlined above and include this additional step.

8. What is $\frac{3}{4}$ of 20 of counters?



How many counters did 1 child get?
How many counters did 3 children get?

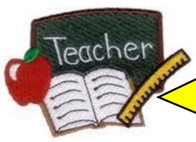
This opportunity can be used to extend the algorithm from step 7. Pupils should be encouraged to deduce that a multiple fraction of a whole number is found by multiplying the answer of a unit fraction by the numerator of the multiple fraction.

$$\frac{1}{4} \text{ of } 20 = 5$$

$$\frac{3}{4} \text{ of } 20 = 5 \times 3$$

Steps 1 to 8 could be applied using a variety of problems across strands. An example for capacity might include:

There were 24 litres of juice at a party. $\frac{5}{8}$ *of the juice was left after the party. How many litres of juice were left? How many litres of juice did the children drink?*



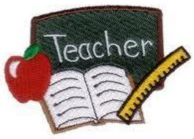
Draw pictures to show how you worked out your answers.
Explain how you got your answer.
So what you did was…
Is there another way of doing it?
Record these using mathematical symbols.

Level B.4

Calculate the number given the unit fraction (with denominators 2,4,8,10)

1. In pairs, pupils work together to solve the following problem. Pupils select appropriate materials to solve the problem, for example, euro coins, paper folding, grid paper, pictures, empty number lines, etc.

   $\frac{1}{10}$ *of my savings is €3. What fraction is my total savings? How much money have I saved altogether?*



How could we do this in a more efficient way?
Did someone do it in a different way than John?
Check if this works for other examples.
Explain what we have discovered.
Mary, can you repeat what John said using your own words?
How can we record the findings using symbols?
Write about what you have learned in your learning log.

2. Repeat this problem using other fractions.
3. Explore and discuss patterns, for example, multiplication facts. This is an ideal opportunity for identifying the algorithm which in this example is $3 \times 10 = 30$.

Level B.5

Calculate the number given the multiple fraction (with denominators 2,4,8,10)

Follow the steps as outlined above and include this additional step.

4. $\frac{3}{10}$ of my savings is €6. How much money have I saved altogether?

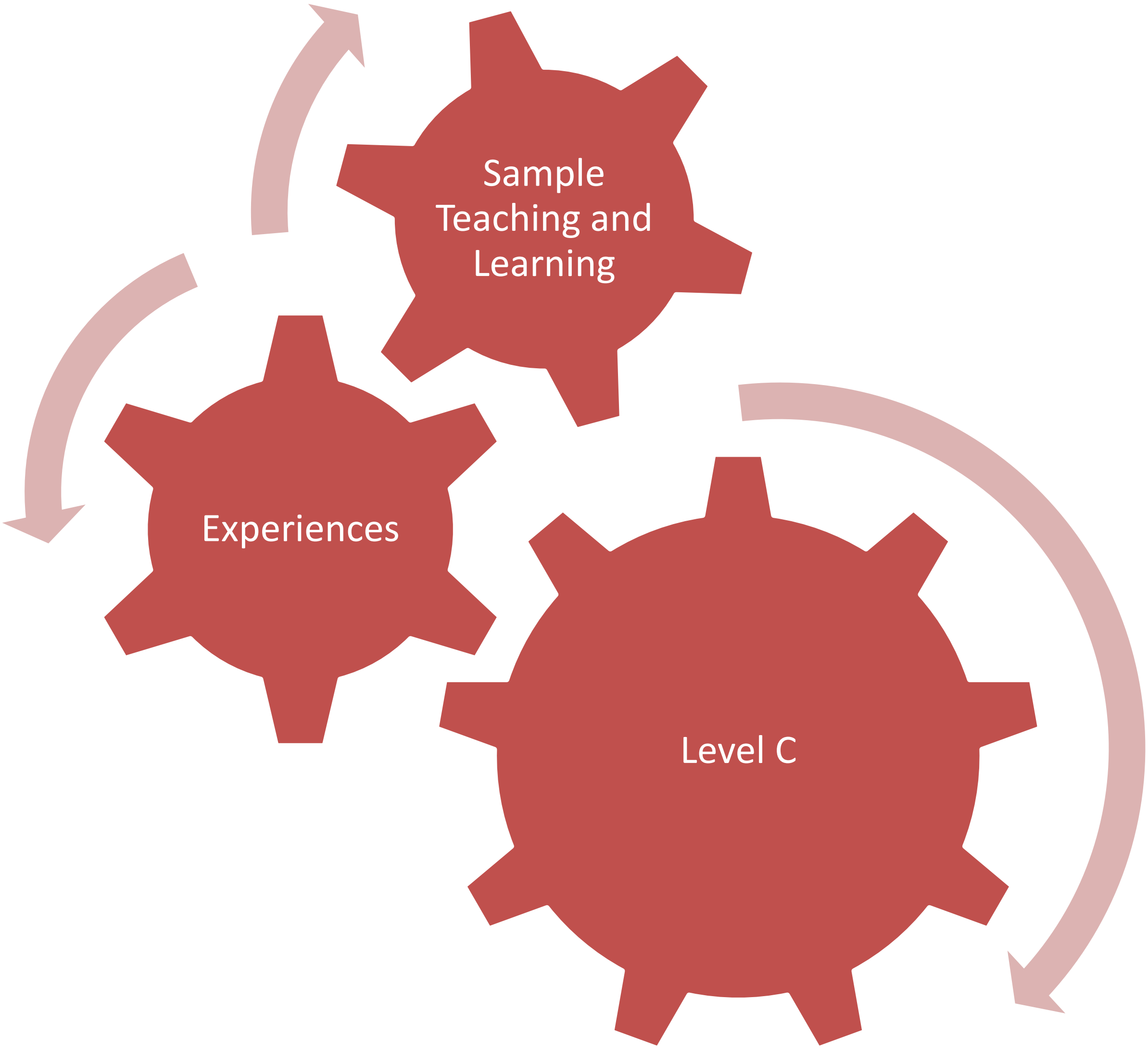
Discuss with your partner what you need to do to solve the problem and estimate what the answer might be.
Draw pictures to show how you worked out your answers.
Explain how you got your answer.
Can you revoice what Mary said?
Did anybody do it in a different way?
Record your answer using mathematical symbols.
Write about what you have learned in your learning log.

This is a difficult concept for pupils. An opportunity can be provided to extend the algorithm from step 3. Pupils should be encouraged to deduce that a whole number can be found by firstly calculating the unit fraction (dividing by the numerator) and then calculating the whole set by multiplying by the denominator

$$\frac{3}{10} = 6$$

$$\frac{1}{10} = 2$$

$$\frac{10}{10} = 20$$


Sample Teaching and Learning
Experiences
Level C

Level C.2

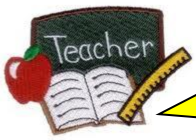
Express improper fractions as mixed numbers and vice versa

A variety of learning experiences should be used to develop the concept of improper fractions and mixed numbers. Pupils will need a lot of opportunities to explore and manipulate concrete materials. The following are some examples of how this might be addressed.

Linear

Unifix Cubes

Teacher needs to assign a unit fraction, for example, $\frac{1}{5}$ to a single cube. For the improper fraction example $\frac{6}{5}$, pupils count out 6 cubes.

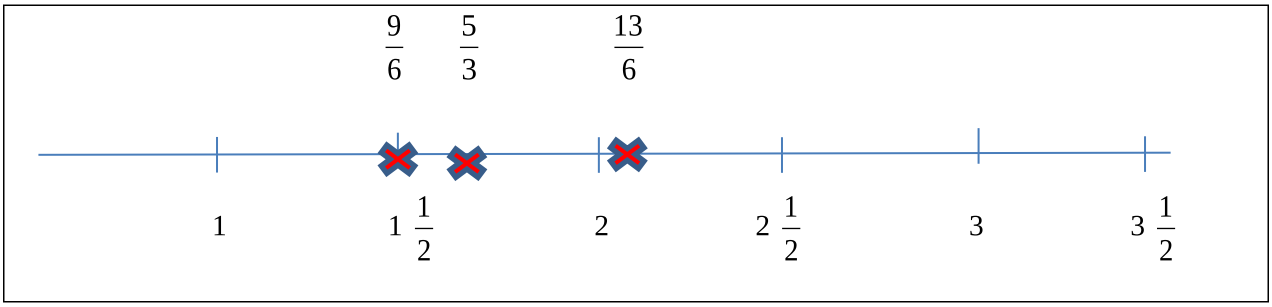


How many fifths do you have?
Is this an unusual type of fraction? Why?
These are called improper fractions.
In pairs, show me several ways of grouping these fifths.
Record your findings.
Do any of these ways represent a whole unit?
How could we represent this improper fraction using symbols?

The above activity should be repeated with other improper fractions, for example, $\frac{5}{3}$, $\frac{9}{6}$, $\frac{13}{6}$.

Benchmarks Number Line

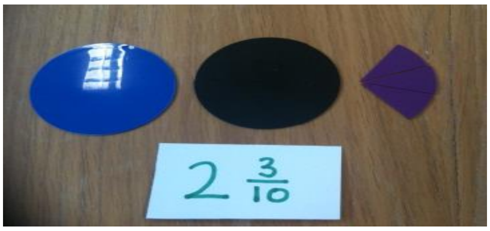
Having completed the above activities, pupils should compare and order these improper fractions by placing them on a benchmarks number line.

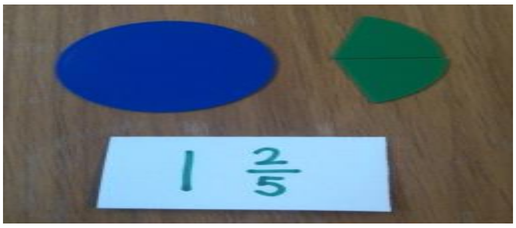


Area Model

Fraction Pieces

Teacher distributes mixed number digit cards to pairs of pupils. Pupils are asked to represent the digit card using fraction pieces. The examples below show $2\frac{3}{10}$ and $1\frac{2}{5}$





Set Model

Drawings

Pupils represent various digit cards (improper fractions and mixed numbers) by drawing their own representation of these. This activity can be further developed by encouraging pupils to first create number stories for these drawings.

Consolidation Activities

Colouring Fraction Wall[20]

The object of this game is to roll dice to create improper fractions. Colour in sections of the fraction walls below that correspond to the fractions found after two rolls.

ICT Opportunities
[Interactive fractions wall]

The Dice

Die A has sides labelled $\frac{}{2}, \frac{}{2}, \frac{}{3}, \frac{}{3}, \frac{}{4}, \frac{}{4}$. This die acts as the denominator.

Die B has sides labelled 7, 6, 5, 4,3, 2. This die acts as the numerator.

Rules of the Game

1. Players take turns rolling both dice. If the player rolls an improper fraction, they colour this fraction equivalent on a wall. Each row on each wall represents one whole. If they do not throw an improper fraction. They miss a turn. For example, if a player throws $\frac{}{2}$ and 3, he or she can colour $1\frac{1}{2}$ sections of any of their three walls. If a player throws $\frac{}{3}$ and 1, he or she misses as turn, as this is not an improper fraction.
2. If a player is unable to use their turn, they must pass.
3. The winner is the first person to have completed all three of their fraction walls.

[20] Adapted from *Colour in Fractions*. from the March 2008 issue of Mathematics teaching in the Middle School.

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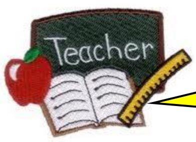
Level C.3

Construct algorithms for equivalent and simplified fractions

Pupils should explore the concept of equivalent fractions using a variety of concrete materials in the context of each model. A variety of fractions should also be used including unit and multiple fractions. Some sample materials are suggested below:



In pairs, represent equivalent fractions in as many ways as possible.
Record these representations using a method of your choice.

Linear Model

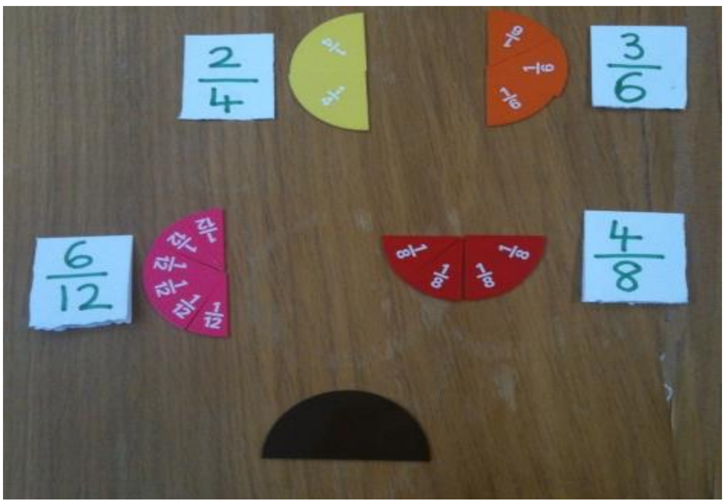
- *Cuisenaire Rods*
- *Paper Strips*
- *Fraction Wall*

Area Model

- *Pattern Blocks*
- *Grid Paper*
- *Fraction Pieces*

ICT Opportunities
[Link: Interactive linear to area model representations]

Equivalent fractions representing a half are shown below using fraction pie pieces.



Set Model

Apples and Bananas[21]

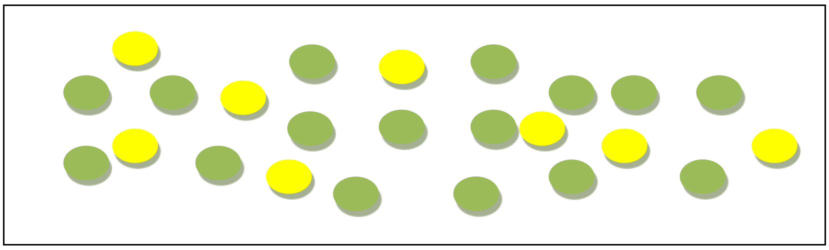
Pupils lay out a specific number of counters. In the following example, 24 counters are used. 16 green counters to represent apples and 8 yellow counters to represent bananas. The 24 make up the whole set. The task is to group the counters into different fractional parts of the whole set and to use the parts to create fractions that are apples and fractions that are bananas.

In what ways did you group your apples and your bananas?
How did you record each grouping?
Did anyone record their grouping in a different way?
Did you discover any equivalent fractions?
Did you spot any patterns?
Could you suggest an algorithm for working out an equivalent fraction?
Test your algorithm to see if it works.
Write about what you have found in your learning log.

[21] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.303

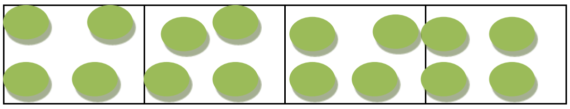
4 counters = 1 whole

16 and 8

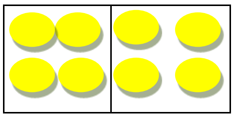


$\frac{16}{24}$ apples $\frac{8}{24}$ bananas

Make the 16 into 4 groups of 4



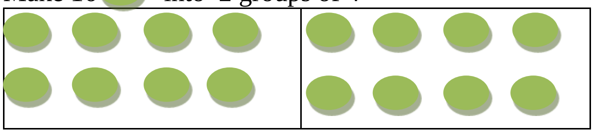
Make the 8 into 2 groups of 4



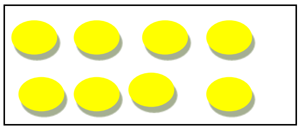
$\frac{4}{6}$ groups are apples

$\frac{2}{6}$ groups are bananas

Make 16 into 2 groups of 4



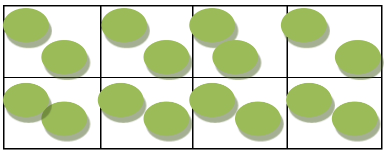
Make 8 into 1 group of 8



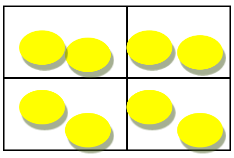
$\frac{2}{3}$ groups are apples

$\frac{1}{3}$ groups are bananas

Make 16 into 8 groups of 2



Make 8 into 4 groups of 2



$\frac{8}{12}$ groups are apples

$\frac{4}{12}$ groups are bananas

Apples are $\frac{16}{24} = \frac{4}{6} = \frac{2}{3} = \frac{8}{12}$

Bananas are $\frac{8}{24} = \frac{2}{6} = \frac{1}{3} = \frac{4}{12}$

Missing Number Equivalencies[22]

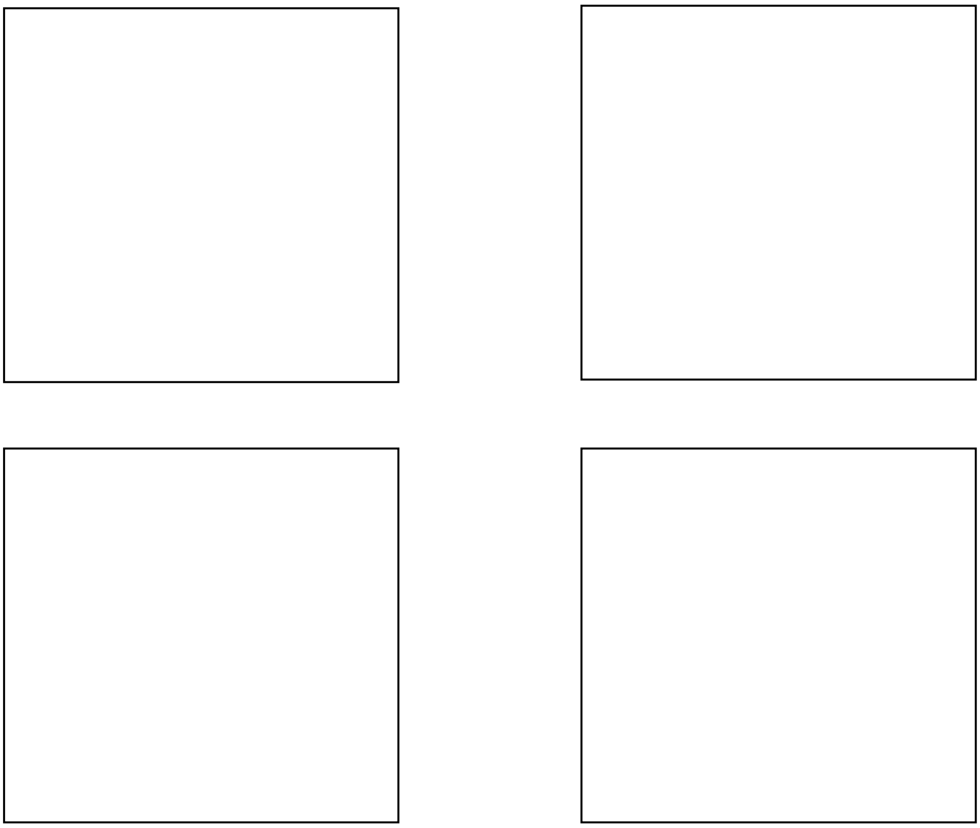
Give the children an equation expressing equivalence between two fractions but with either a numerator or denominator missing. Ask them to draw a picture to solve.

$\frac{5}{3} = \frac{\square}{6}$ $\qquad$ $\frac{9}{12} = \frac{3}{\square}$

Teacher may specify a particular model within which to work, for example, an area or set model. Alternatively, pupils may select a model themselves. They should explore the problem using the concrete model and then record their findings pictorially.

Developing an Equivalent-Fraction Algorithm[23]

Slicing Squares

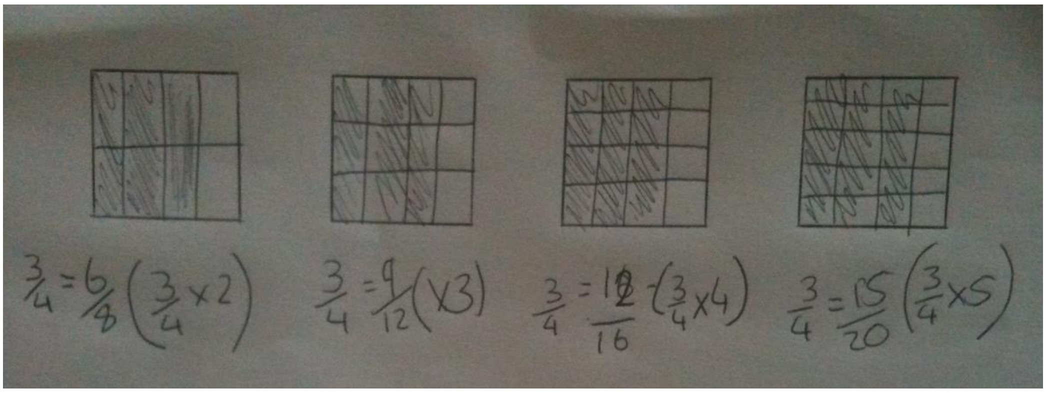


Ask each pupil to complete the following task:

1. Using **vertical** dividing lines slice each square into quarters.

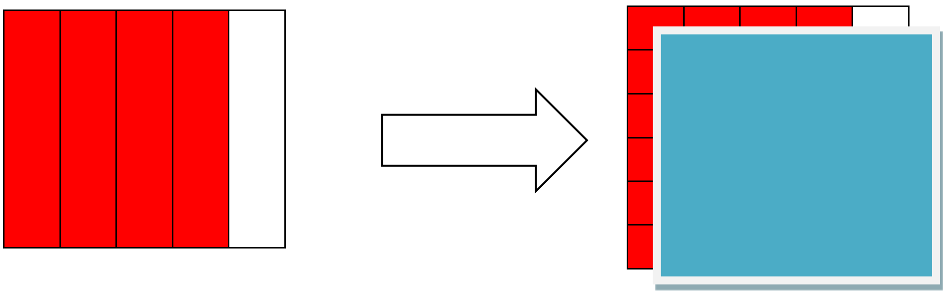
[22] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.304
[23] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.304, 305

2. Now, each square must be divided **horizontally** into equal-sized slices. But each square must be partitioned differently. Use one to eight horizontal slices.
3. Now, record an equation showing the equivalent fractions for each sliced square.
4. Ask children to examine their drawings and equations to look for any patterns.



5. This activity can be repeated with four more squares, using a different fraction.

After this activity, teacher writes on the board the equations for four or five different fraction names found by the pupils. Discuss patterns. To focus the discussion, show a square illustrating $\frac{4}{5}$ made with vertical lines.



$\frac{4}{5}$ $\frac{4}{5}=?$

Slice the square into six parts in the opposite direction (horizontally). Cover all but two edges of the square as shown above. Ask, ‘What is the new name for my $\frac{4}{5}$?’ (24/30)

The reason for this exercise is that many pupils simply count the small regions and never think to use multiplication. With the covered square, students can see that there are four columns and six rows to the shaded part, so there must be 4 x 6 parts shaded. Similarly there must be 5 x 6 parts in the whole.

Therefore, the new name for $\frac{4}{5}$ is $\frac{4}{5} \times \frac{6}{6}$ or $\frac{24}{30}$

Using this idea, the pupils can return to the fractions on their worksheet to see if the pattern works for other fractions.

Examine examples of equivalent fractions that have been generated with other models, and see if the rule of multiplying top and bottom numbers by the same number works in those contexts also.

Writing Fractions in Simplest Terms

Finding equivalent fractions results in fractions with larger denominators. Simplifying fractions involves renaming the fraction so that the numerator and denominator have no common whole-number factors. It is sometimes referred to as expressing a fraction in its 'lowest terms'. The algorithm for simplifying fractions should obviously be connected back to a reversal of the equivalence algorithm.

COMMON LANGUAGE ERROR

The phrase 'reducing fractions' when simplifying fractions should not be used because it implies that the fraction is being made smaller. This is not the case. Fractions are simplified not reduced.

Consolidation Activity

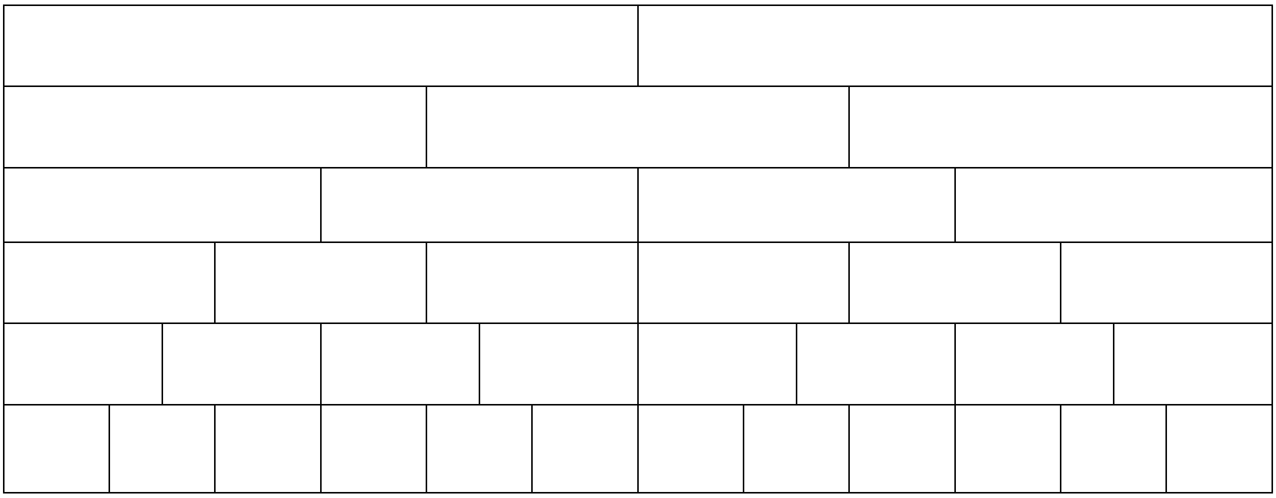
Colour in Fractions. (From the March 2008 issue of *Mathematics teaching in the Middle School)*

The object of this game is to roll dice to create fractions up to twelfths. Colour in sections of the fraction wall below that correspond to the fractions found after two rolls.

The Dice

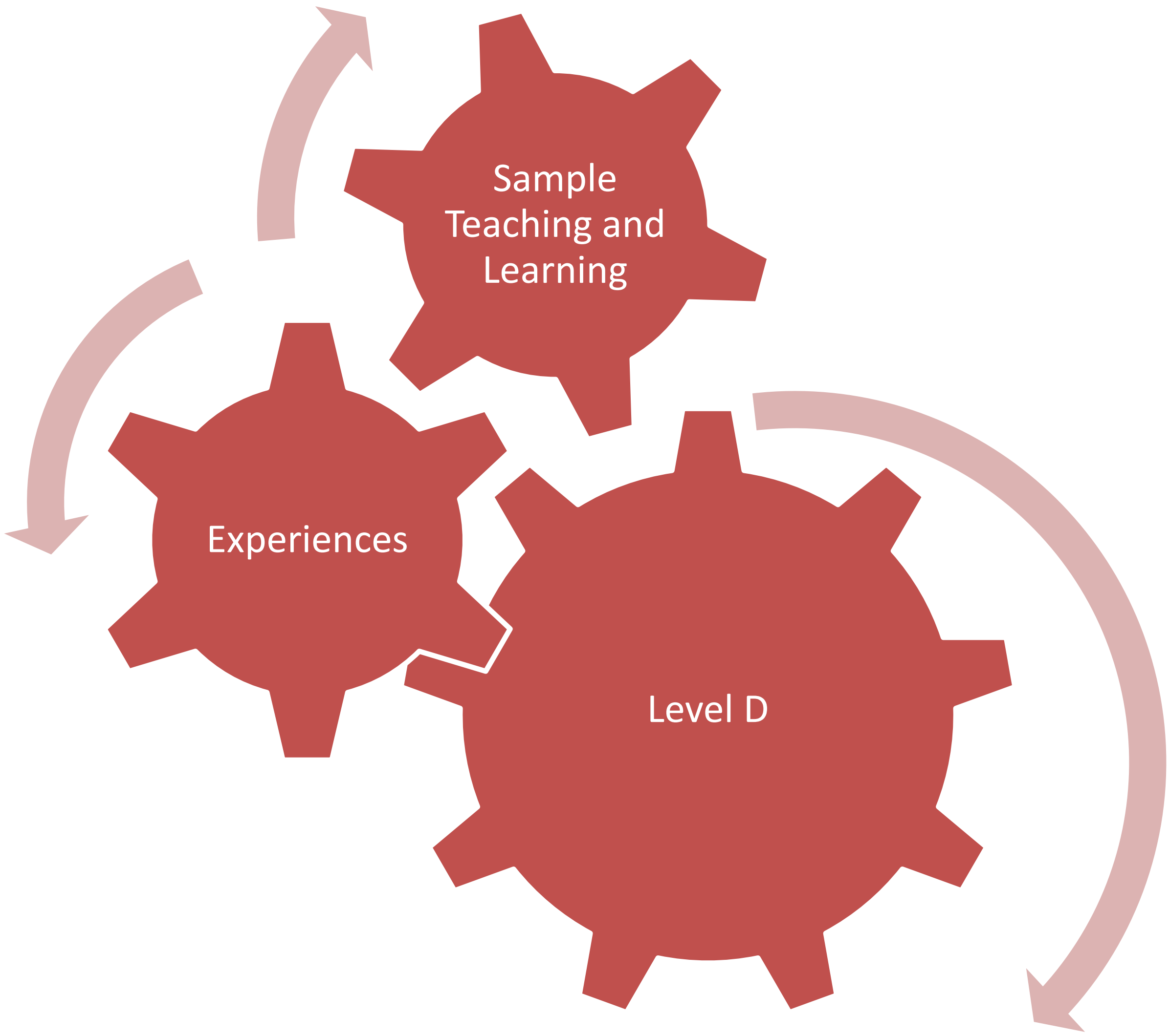
Die A has sides labelled 1, 2, 2, 3, 3, 4. This die acts as the numerator.

Die B has sides labelled $\frac{\ }{2}, \frac{\ }{3}, \frac{\ }{4}, \frac{\ }{6}, \frac{\ }{8}, \frac{\ }{12}$. This die acts as the denominator.



Rules of the Game

1. Players take turns rolling both dice. Each player will make a fraction. Each row on the wall represents one whole.
2. Each player colours the fraction equivalent on their wall. For example, if a player throws 2 and $\frac{\ }{4}$, he or she can colour: $\frac{2}{4}$ of one line; $\frac{4}{8}$ of one line; $\frac{1}{4}$ of one line and $\frac{2}{8}$ of another line; any other combination equivalent to $\frac{2}{4}$.
3. If players are unable to use their turn, they must 'pass'.
4. The winner is the first person to have completed their entire wall.


Sample Teaching and Learning
Experiences
Level D

Level D.1

Add and subtract fractions (first with like denominators, then repeat with unlike denominators

Like Denominators

If pupils have experienced lots of activities involving counting fractions, they should be able to add and subtract fractions with like denominators without difficulty. Pupils should explore addition and subtraction of fractions with like denominators using all three models (linear, area, set).

Unlike Denominators

As with all number computation, estimation is fundamentally important. There are a number of ways of estimating which may be helpful before adding and subtracting fractions:

- estimating the sum or difference of two fractions in terms of whether the answer will be greater than or less than 1
- estimating the sum or difference of two fractions in terms of whether the answer will be more or less than $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2 or 3.[24]

The discussion following these estimation exercises should then try to encourage pupils to think if their exact answer is more or less than the estimate they gave. The following examples may be useful when asking pupils to estimate:

a. $\frac{1}{8} + \frac{4}{5}$

b. $\frac{9}{10} + \frac{7}{8}$

c. $\frac{3}{5} + \frac{3}{4}$

d. $\frac{3}{4} - \frac{1}{3}$

[24] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.311

e. $\frac{11}{12} - \frac{3}{4}$

f. $1\frac{1}{2}$ - $\frac{9}{10}$.

Asking pupils to record their estimate on an empty number line is another effective strategy.

AREA MODELS

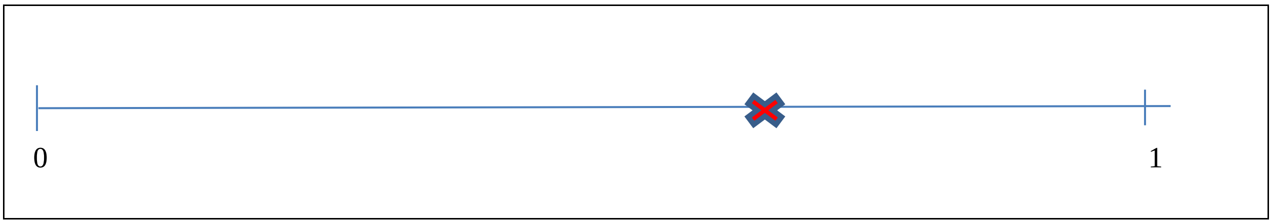
Research suggests that circular area models are the most effective for developing the concepts of addition and subtraction of fractions. (Cramer, Wyberg & Leavitt, 2008)

Area Model

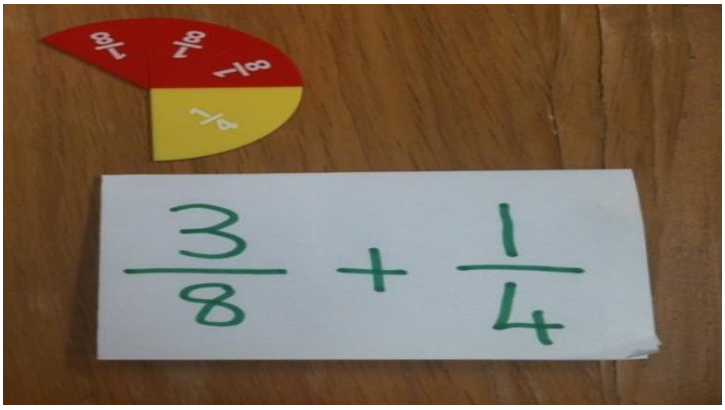
Fraction pie pieces are useful when adding and subtracting fractions with unlike denominators. The following is one possible approach that pupils might take to solve the following problem.

Two brothers ate some pizza. John ate ¼ of the pizza and Mark ate $\frac{3}{8}$*. How much of the pizza did they eat altogether?*

1. The pupil records their estimate on an empty number line

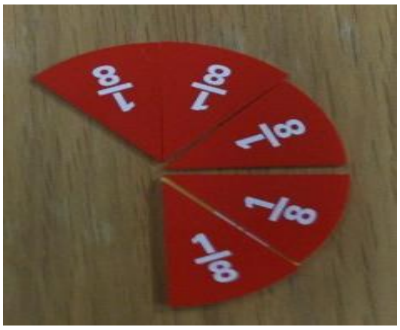


2. The pupil combines fraction pie pieces to represent the problem.



Can you write your answer as a fraction? Explain.
What can we do next?

3. One approach that a pupil may take is to exchange $\frac{1}{4}$ for two $\frac{1}{8}$ pieces.



Now write your answer as a fraction.
Explain why it is easier to do this now.
Write about this in your learning log.

4. Pupils should carry out similar activities using different fraction pieces, for example, $\frac{1}{3} + \frac{3}{6}$;

$\frac{2}{5} + \frac{4}{10}$; $\frac{1}{4} + \frac{5}{12}$.

5. An alternative approach a pupil may attempt to solve the problem might involve building up to the whole. This is sometimes referred to as 'Residual Thinking'[25]. Here the pupil realises that three more eighths are required to fill the whole. They then deduce that

$\frac{3}{8} + \frac{1}{4} = \frac{5}{8}$ as $\frac{5}{8}$ is $\frac{3}{8}$ away from the whole or $\frac{8}{8}$.

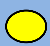
[25] Cramer, Wyberg, Leavitt (2008)

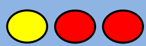
6. Pupils should then identify patterns leading to a discussion regarding the need for a common denominator.

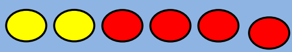
ERROR ANALYSIS

The most common error when adding fractions is to add both numerators and denominators. These errors should be discussed at a whole-class level when they occur.

Discuss the following 'solution' for adding $\frac{1}{2} + \frac{1}{3}$ in terms of whether the pupil could be correct or not. Also discuss why.

$\frac{1}{2}$ 

$\frac{1}{3}$ 

Add 

Therefore $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ (Add numerators and denominators)

(Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.316)

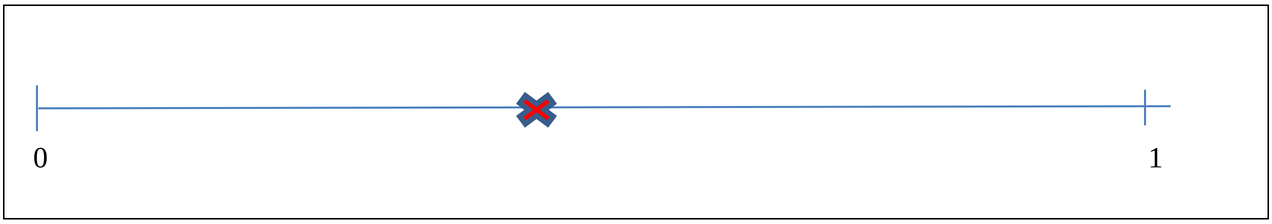
ICT Opportunities

[Link: Adding fractions]

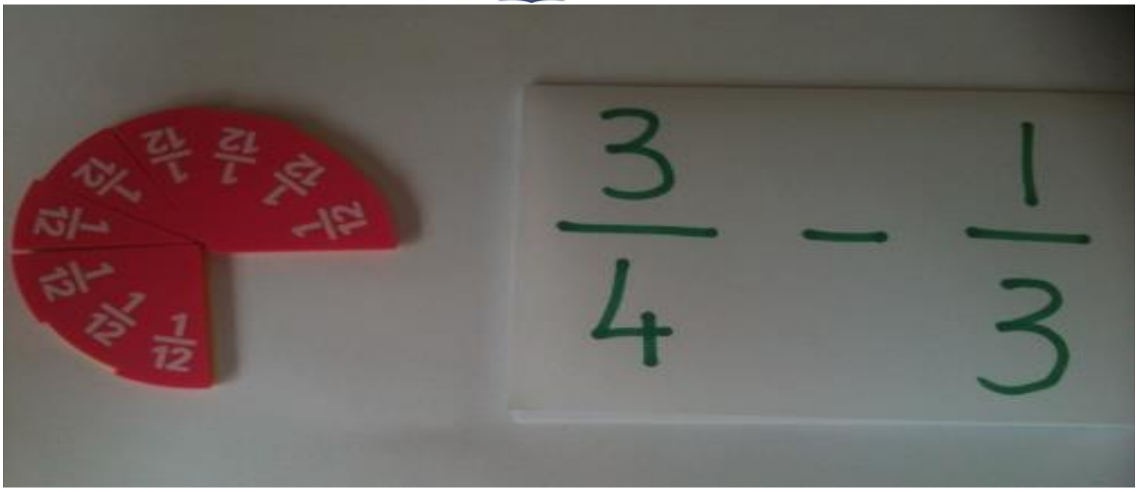
An example of a subtraction problem might be:

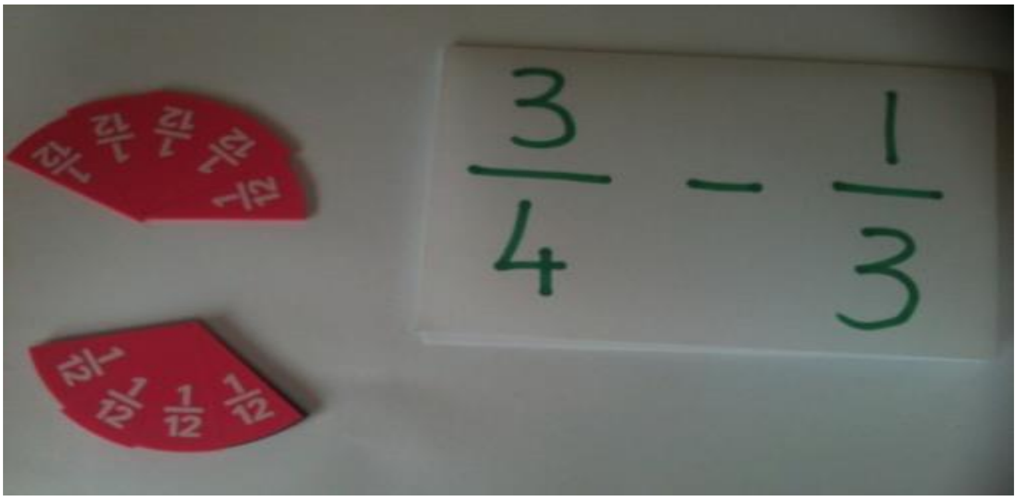
The shopkeeper has $\frac{3}{4}$ *of a metre of red ribbon. Caoimhe needs* $\frac{1}{3}$ *of a metre of red ribbon. After Caoimhe buys the red ribbon, how much red ribbon has the shopkeeper left?*

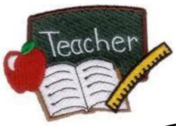
1. Again, the pupils record their estimates on an empty number line.



2. Using either concrete materials or drawings, the pupils attempt to illustrate the problem. One possible approach pupils might take is outlined below:







What did you find difficult about solving this problem?
Why did you exchange for twelfths?
Did anybody try to solve the problem in a different way?
How did your answer compare to your estimate?
Why do you think a common denominator is required?

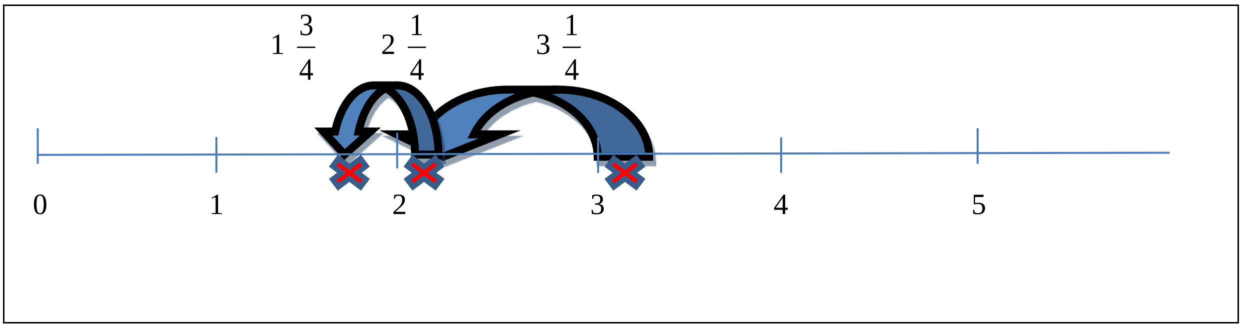
3. The need for a common denominator should arise from discussions in completing these types of activities.

ICT Opportunities
Link: Subtracting fractions puzzle

Developing an algorithm for addition and subtraction using the Common Denominator approach:

In order to use the algorithm for adding fractions, common denominators are required. As pupils engage in concrete and pictorial problem-solving activities requiring the addition and subtraction of fractions, they will discover the need for a common denominator. This is due to the fact that the algorithm only works successfully when adding or subtracting fraction parts that are the same size. However, pupils who have strong fraction number sense will often be able to add and subtract fractions with unlike denominators without ever getting a common denominator. The number line is an excellent tool through which fractions can be added or subtracted mentally.

$$\mathbf{3\frac{1}{4} - 1\frac{1}{2}}$$



Here the pupil has jumped back 1 from $3\frac{1}{4}$, arriving at $2\frac{1}{4}$ and then jumped back a further $\frac{1}{2}$ to arrive at the answer $1\frac{3}{4}$.

If pupils have developed deep understandings of the concept of fraction equivalence, then they should more easily see common relationships between various denominators. Continued reinforcement of the equivalence concept is necessary whilst the pupils are engaged in the addition and subtraction of fractions.

Common Denominators

Pupils may have difficulty finding common denominators because they are not able to work out a common multiple for both denominators. This may be due to limitations in their knowledge of multiplication facts. It may be necessary to drill pupils in order to achieve a good command of these facts.

The following drill activity may assist these pupils:

Common Denominator Flash cards:[26]

Make flash cards with various pairs of denominators between 2 and 12. For each card, pupils try to give the lowest multiple that is common to both denominators. For example, if the pupil turns over a 3, 4 card, the multiple will be 12. It is important to include:

- pairs of denominators that are prime, for example, 3 and 7
- pairs in which one is a multiple of the other, for example 2 and 6 and
- pairs that have a common divisor , for example 6 and 12.

ICT Opportunities

Link: Exploring common denominators

[26] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.317

Level D.2

Add and subtract mixed numbers

A new algorithm for addition and subtraction of mixed numbers is not necessary. Including mixed numbers in the activities in Level D.1 will allow pupils to draw this conclusion. Generally, pupils will add or subtract whole numbers first and then add or subtract the fractions. In the most straight forward operations, this is all that will be required. However, in certain operations, there will be a need to regroup across the whole number and fraction. For example, $1\frac{7}{8} + 3\frac{3}{4}$ or $5\frac{1}{3} - 2\frac{3}{4}$

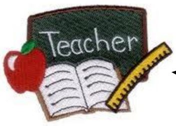
Pupils should continue to explore possible different approaches to solving these types of problems. The following strategies may be used by pupils in solving a problem such as this:

There were 3 $\frac{2}{5}$ litres of milk in the fridge. Seán drank 1 $\frac{4}{5}$ litres. How much milk was left?

1. Before attempting to solve the problem, pupils should record their estimate on a number line.
2. Generally, it is advisable to subtract the whole numbers first.
3. **Possible approach A**: having subtracted the whole numbers, the pupil may take $\frac{4}{5}$ from the remaining whole of 2, leaving $1\frac{1}{5}$. They will then add on $\frac{2}{5}$ to give an answer of $1\frac{3}{5}$.
4. **Possible approach B:** having subtracted the whole numbers, the pupil may leave the $\frac{2}{5}$ and only subtract $\frac{2}{5}$ rather than $\frac{4}{5}$ from 2.
5. **Possible approach C**: this involves exchanging a whole for $\frac{5}{5}$ and adding it to the $\frac{2}{5}$, thus renaming $3\frac{2}{5}$ as $2\frac{7}{5}$. $1\frac{4}{5}$ can easily then be subtracted.

6. **Possible approach D**: pupils use their algorithm for converting mixed numbers to improper fractions and make then make the calculation. $\frac{17}{5} - \frac{9}{5} = \frac{8}{5}$. The pupil can then convert this improper fraction back into a mixed number, using their algorithm.

7. As pupils work through various approaches, it is imperative that they share their mathematical ideas with each other.



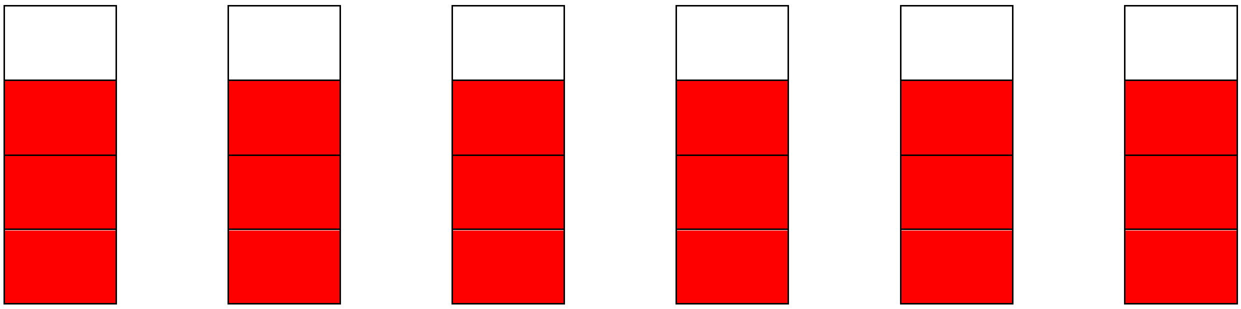
How did you go about solving this problem?
Has anyone a different approach for solving the problem?
John, in your own words can you tell me Mary's approach?
How did you record your findings?
Did anyone record their findings in a different way?
Record what you have learned in your learning log.

Level D.3

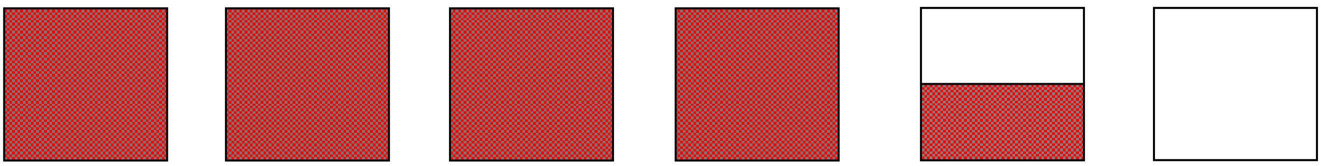
Multiply a fraction by a whole number

Multiplying Fractions

Teacher questioning is vitally important in order to stimulate and deepen pupils' understanding of multiplication in fractions. Rather than relying on a rule, it is more beneficial for pupils to think about the concept of multiplying fractions – what is actually taking place. The following example illuminates what it means to multiply fractions. The commutative rule of multiplication means that $\frac{3}{4} \times 6$ may be interpreted in two ways: '6 lots of $\frac{3}{4}$' or 'three-quarters of 6' and so pupils may find visual representations of these helpful in order to contextualise this.

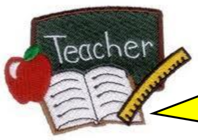


'6 lots of $\frac{3}{4}$'



'three-quarters of 6'

Teacher questioning can aid pupils' understanding here.



What does $\frac{3}{4}$ of 6 look like?
How could we draw a representation of this? or What materials could we use to represent this?
Will the answer be bigger or smaller than 6?
Will the answer be much smaller than 6?

Similar questions can be used to explore the alternative interpretation of this sum which is '6 lots of $\frac{3}{4}$'. Pupils need a lot of experience using concrete materials and drawing in order to represent these sums so that the concept and the rationale of what is happening when fractions are multiplied can be developed. This experience is particularly important because the answers which pupils normally get when multiplying whole numbers are always bigger than the numbers in the sum whereas they are smaller when multiplying fractions. The concept of multiplying fractions is therefore counter-intuitive for pupils and will require a lot of hands-on experience in addition to lots of discussion.

Repeated Addition

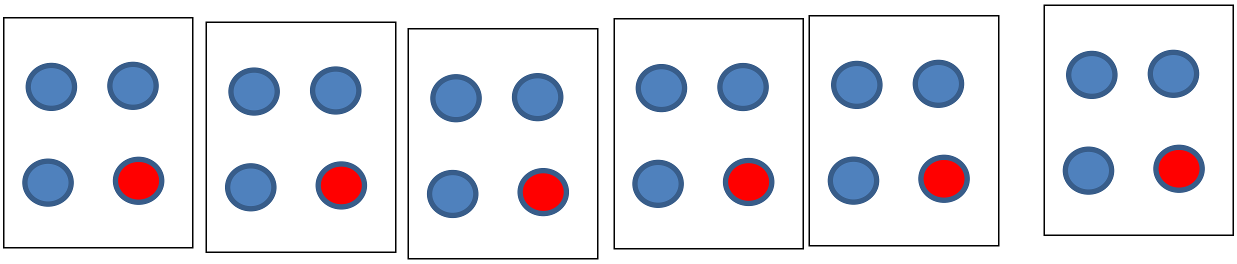
Repeated addition builds on pupils' counting skills and can be used as a grounded entry point for exploring the multiplication of fractions. See counting activities in the Teaching and Learning Experiences section of Level B.1.

Set Model

Collections of objects can be used to explore the concept of multiplication. For example, $\frac{3}{4} \times 6$ could be represented in two ways:

1. $\frac{3}{4}$ of 6 counters cannot be found by sharing because the counters cannot be split. It may be useful to use some materials which can be split or cut, for example, a collection of straws or pipe cleaners can be used because they can be shared equally by cutting them.

2. 6 lots of $\frac{3}{4}$ can be represented using a variety of sets, for example, different coloured counters can be used:

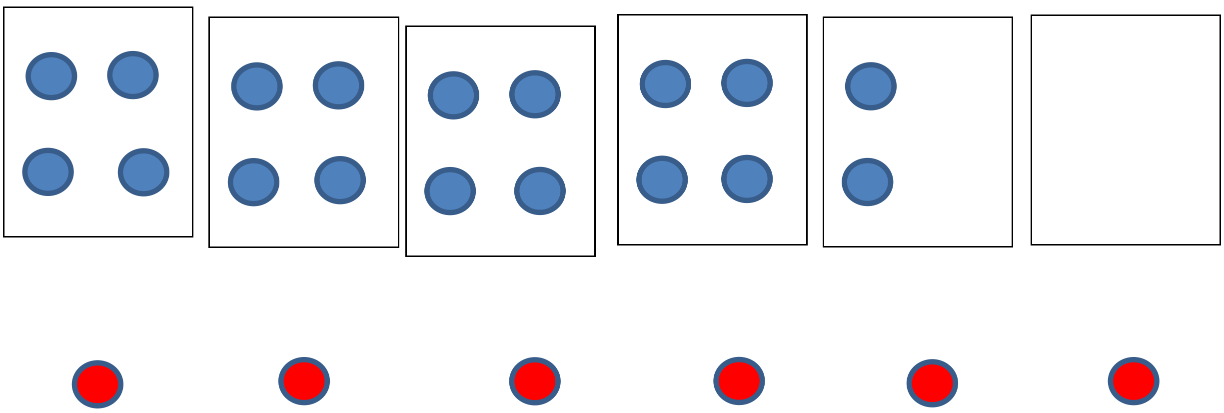


The next step is to encourage pupils sort these sets in order to determine what 6 lots of $\frac{3}{4}$ are equal to.

Could we tidy up these sets so that we can find out what '6 lots of $\frac{3}{4}$' are equal to?
How could we do that?
In pairs, try sorting these sets and then discuss what you did.
How did you sort the sets?

Some pupils may decide to sort for colour that is, putting the blues together to make full sets of blues.

The pupils then realise that they have 4 full sets of blue counters and one half set of blue counters so '6 lots of $\frac{3}{4}$' is equal to $4\frac{1}{2}$.

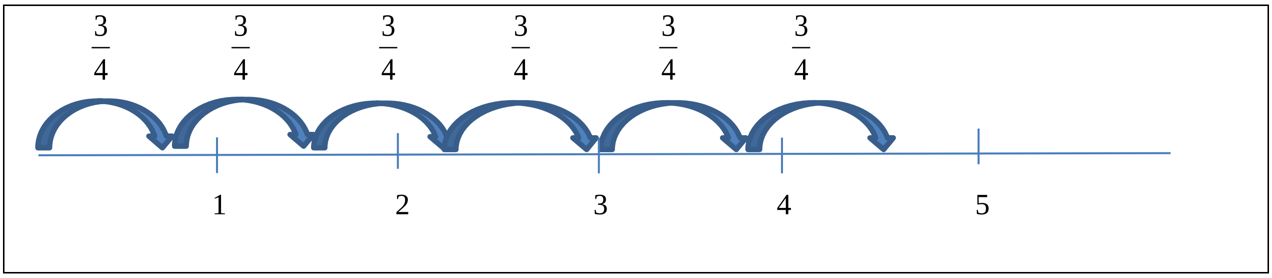


Linear Model

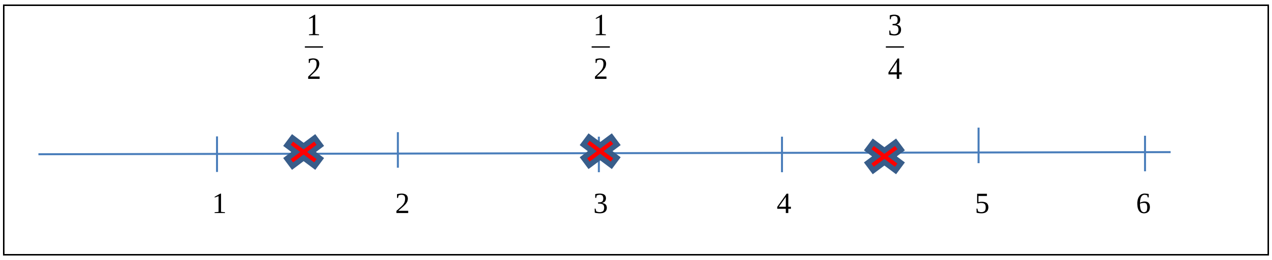
Empty Number Lines

Empty number lines can also be used to contextualise what is happening when multiplying fractions.

The pupil draws a straight line, puts in numbers and then makes the necessary amount of ‘jumps’. For example, ‘6 lots of $\frac{3}{4}$’ could be represented:



Similarly, $\frac{3}{4}$ of 6 could be represented like this:



These experiences will make the algorithm more meaningful when it is introduced.

Sample Problems

Ask pupils to use any materials or drawings to figure out the answer to the following tasks.

Linear Model

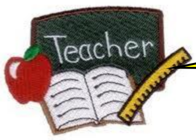
The walk from school to the shop takes 15 minutes. When Evie asked her mum how far they had gone, her mum said they had gone $\frac{2}{3}$ of the way. How many minutes have they walked?

Area Model

Seán filled 15 glasses with $\frac{2}{3}$ cup of water each. How much water did Seán use?

Set Model

There are 15 cars in Tom's car collection. $\frac{2}{3}$ of the cars are red. How many red cars does Tom have?



We will need to tidy our material away soon. Can we find a way of saving our ideas in another way?

In the linear and set model pupils might partition 15 into three parts, and then see how many are in two parts. Recording it in symbols $\frac{2}{3}$ of 15 gives the following result 15÷3×2. In the area model the problem is 15 groups of $\frac{2}{3}$ and not $\frac{2}{3}$ of a group of 15. Although the commutative property means that these numbers can be switched it is important that pupils understand each type of representation when their meanings are different. This problem might be solved in a repeated addition strategy, for example, $\frac{2}{3} + \frac{2}{3} + \ldots\ldots \frac{2}{3}$ = 30/3 = 10. Pupils may also notice that they multiplied the numerator by 15 and divided by 3.

Level D.4
Multiply a fraction by a fraction

Ask pupils to use any materials or drawings to figure out the answer to the following tasks.

Linear Model

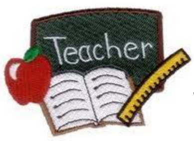
Someone ate $\frac{1}{10}$ of a french stick . If you use $\frac{2}{3}$ of what is left of the french stick to make toast, how much of the whole french stick will you have used?

Area Model

You have $\frac{3}{4}$ of a pizza. If you gave $\frac{1}{3}$ of the left over pizza to your brother, how much pizza will your brother get?

Set Model

Annie lost $\frac{1}{3}$ of her buttons. She gave her sister $\frac{3}{5}$ of the buttons that she had left. How many buttons did her sister have?



We will need to tidy our material away soon. Can we find a way of saving our ideas in another way?

Standard Algorithm[27]

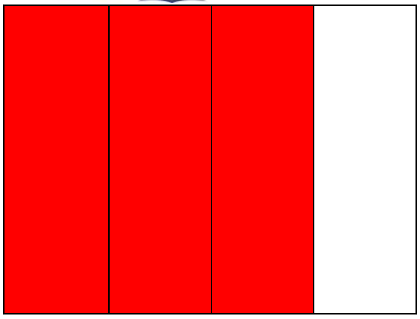
The area model is useful for connecting to the standard algorithm for multiplying fractions, for example, the following activity could be used to find $\frac{3}{5} \times \frac{3}{4}$ and then to link this exploration with the algorithm.

1. Give pupils a square on grid paper. Ask them to illustrate the first fraction. In the above question the pupils are required to find $\frac{3}{5}$ of $\frac{3}{4}$, they first must show $\frac{3}{4}$.

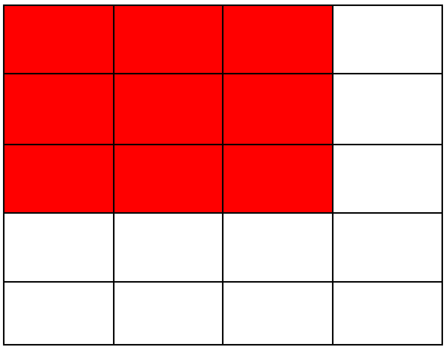
[27] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.320

ICT Opportunities

[Link: Interactive multiplying a fraction by a fraction]



2. To find fifths of the $\frac{3}{4}$, draw four horizontal lines through the three quarters or all across the square so that the whole area is divided into the same size partitions. Then shade in $\frac{3}{5}$ of the $\frac{3}{4}$.



$$\frac{3}{5} \quad \frac{3}{4} = \frac{\text{number of parts in a product}}{\text{kinds of parts}} \quad \frac{= 3 \times 3 = 9}{5 \times 4 = 20}$$

Now do the same activity for the following:

(1) $\frac{5}{6}$ x $\frac{1}{2}$ (2) $\frac{3}{4}$ x $\frac{1}{5}$ (3) $\frac{1}{3}$ x $\frac{9}{10}$

How did you figure out what the unit (denominator) of the fraction was?
In task number (1) how did you figure out that the denominator would be twelfths?
Is this a pattern that is true for the other examples?
Can you find a pattern for how the number of parts (numerator) is found?
Write about your findings in your learning log.

Level D.5

Divide a whole number by a unit fraction

Misconceptions[28] when teaching division of fractions

Thinking the answer should be smaller

Based on pupils' experience with whole number division, pupils may think that when dividing by a fraction, the answer should be smaller. This is true if the divisor is a fraction greater than one, for example, $\frac{4}{3}$ but is not true if the fraction is less than one. Estimation is critical here. It will help pupils decide whether the answer is reasonable.

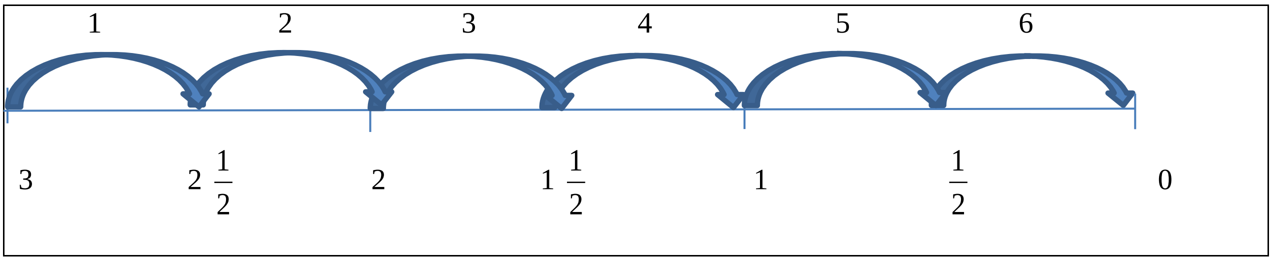
Connecting the illustration with the answer

Pupils may understand that $1\frac{1}{2} \div \frac{1}{4}$ means how many $\frac{1}{4}$ in $1\frac{1}{2}$, so they count how many $\frac{1}{4}$ and get 6. But in recording their answer they can confuse the fact that they were using fractions and record their answer as $\frac{6}{4}$.

Repeated Subtraction

Dividing a whole number by a unit fraction is the same as repeated subtraction or equal groups, for example:

If you have 3litres of milk how many bottles holding $\frac{1}{2}$ *litre can you fill?*



Starting at 3 on the number line, pupils can jump back $\frac{1}{2}$ from each bottle of milk until they reach 0.

[28] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009)

Ask pupils to use any manipulative or drawing to figure out the answer to the next question:

You are having a birthday party. You have ordered 6 kilogrammes of sweets. If you serve $\frac{1}{4}$ of a kilogramme of sweets to each guest, how many guests can be served?

We will need to tidy our material away soon.
Can we find a way of saving our ideas in another way?
John, could you share your idea with all of us?

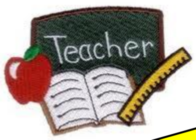
The introduction of an empty number line here can be useful to represent division as repeated subtraction.

Other problems may include:

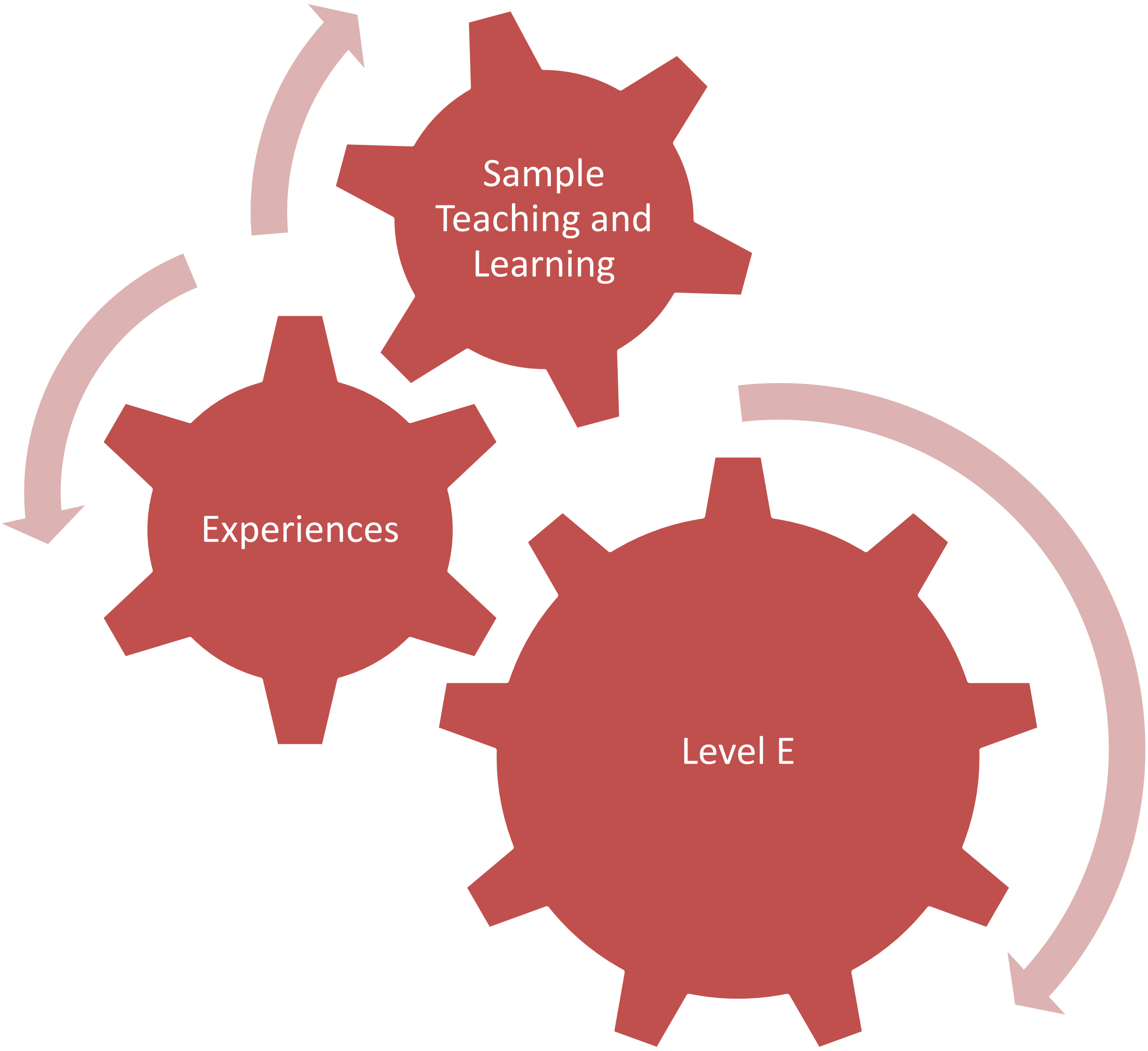
A serving is a $\frac{1}{2}$ of a cookie. How many servings can I make from 2 cookies?

A serving is $\frac{1}{3}$ of a cookie. How many can I make from 3 cookies?

A serving is $\frac{1}{5}$ of a cookie. How many can I make from 5 cookies?



How did you work out your answer?
Did anybody do it differently?
Mary, can you put John's idea into your own words?
How could you represent your findings in another way?
What patterns did you notice?
Test your hypothesis using other problems.


Sample Teaching and Learning
Experiences
Level E

Level E.1
Understand and use simple ratio

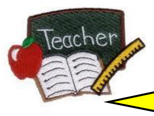
The following explanations of ratio may be useful:[29]

1. A ratio is a multiplicative comparison of two quantities or measures.
2. Ratios and proportions involve **multiplicative** rather than **additive** comparisons. Equal ratios result from multiplication or division, not from addition or subtraction.
3. Proportional thinking is developed through activities involving comparing and determining the equivalence of ratios and solving proportions in a wide variety of problem-based contexts and situations without recourse to rules or formulas.
4. Part-whole relationships (fractions) are an example of a ratio. Fractions are also one of the principal methods of representing ratios.

Additive versus Multiplicative Reasoning

A fortnight ago, two flowers were measured at 8cm and 12 cm respectively. Today they are 11cm and 15cm tall. Did the 8cm or 12cm flower grow more?[30]

1. Ask pupils to discuss this problem, to suggest a solution and to give a reason for their answer.
2. Note: there are two possible solutions to the problem. The first uses additive reasoning, that is that a single quantity was added to each measure to result in two new measures. 8 + 3 =11 and 12 + 3 = 15. The solution using this form of reasoning is that both flowers grew the same amount. The second solution involves the use of multiplicative reasoning, whereby the growth of each flower is compared to the original height of the flower. The first flower grew $\frac{3}{8}$ of its height while the second grew $\frac{3}{12}$. Based on this multiplicative view ($\frac{3}{8}$ times as much more) the first flower grew more. Bothe the additive and multiplicative approach produce valid solutions.
3. Discussion should focus on the comparison between the additive and multiplicative approach to build children's understanding of proportional reasoning..



How did you solve the problem?
Did anyone get a different solution?
Could we represent the amount each flower grew as a fraction of its original height?
How do these fractions compare?

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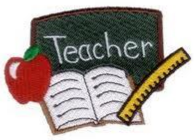
[29] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.348
[30] Adapted from Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.351

Identifying Multiplicative Relationships[31]

There are 16 pupils in sixth class and 12 say that they are football fans. The remaining pupils are not football fans.

1. Ask pupils to describe any relationships they can between the pupils who are football fans and those who are not. Establish that there are 4 non-football fans.
2. Note: the three possible relationships are:
   - There are 8 more football fans than non-football fans.
   - There are 3 times as many fans as non-fans.
   - For every 3 pupils who like football, there is 1 who does not.

ICT Opportunities
[Link: Ratio and proportion problems]

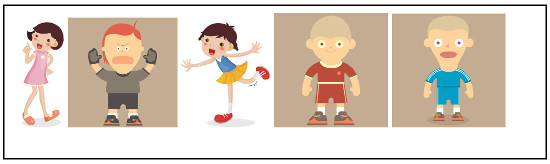


Share one of the relationships you discovered with the class.
Did anyone find another relationship?
What operation did you use to work out the relationship?(Adding or multiplying)
Let's examine the multiplicative relationship. Can you suggest a way of recording this? (3:1)

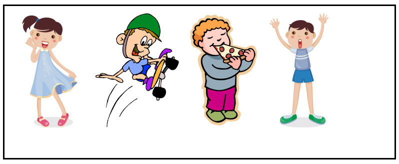
Comparing Ratios Activities[32]

1. Which team has more girls?

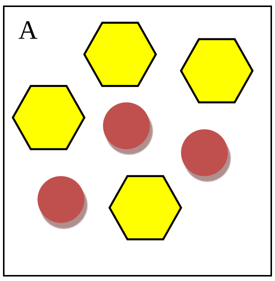
The Stars

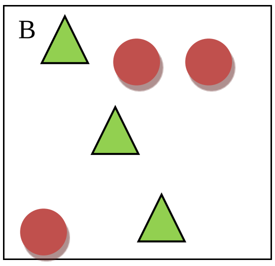


The Comets



2. Which set has more circles?





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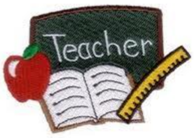
[31] Adapted from Cai and Sun (2002) p.196
[32] Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.352



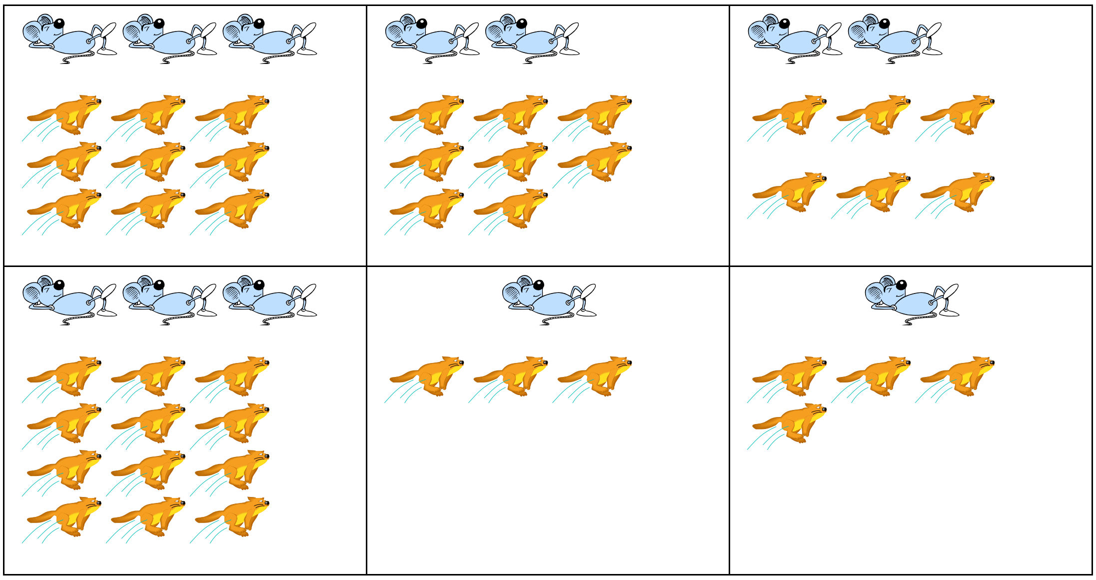
Which team/set has more girls/circles?
Could you look at these problems in another way?
Could fractions help you?
What does that fraction tell you?
Could you express that fraction as a ratio?
How do the fractions compare with each other?
Summarise what you have learned with your partner and write this in your learning log.

Pupils should experience a variety of these types of comparing activities with plenty of discussion opportunities.

Equivalent and Different Ratios: The pupil's understandings of the equivalence concept in relation to fractions will greatly influence their understanding of equivalent ratios. The following comparison activities will help to develop the pupil's understanding of equivalent ratios. Note: it is very useful to include pairs of ratios that are not proportional but have a common difference. For example 5:8 and 9:12 are not equivalent ratios but the common differences are the same 8 – 5 = 12 – 9. Pupils who focus on the additive relationship are not seeing the multiplicative relationship of proportional reasoning.. The following activities[33] may be useful:

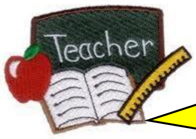
Different Objects, Same Ratios

Prepare cards that illustrate various ratios of two different objects, for example, apples to fruit bowls, mice to cats, etc.



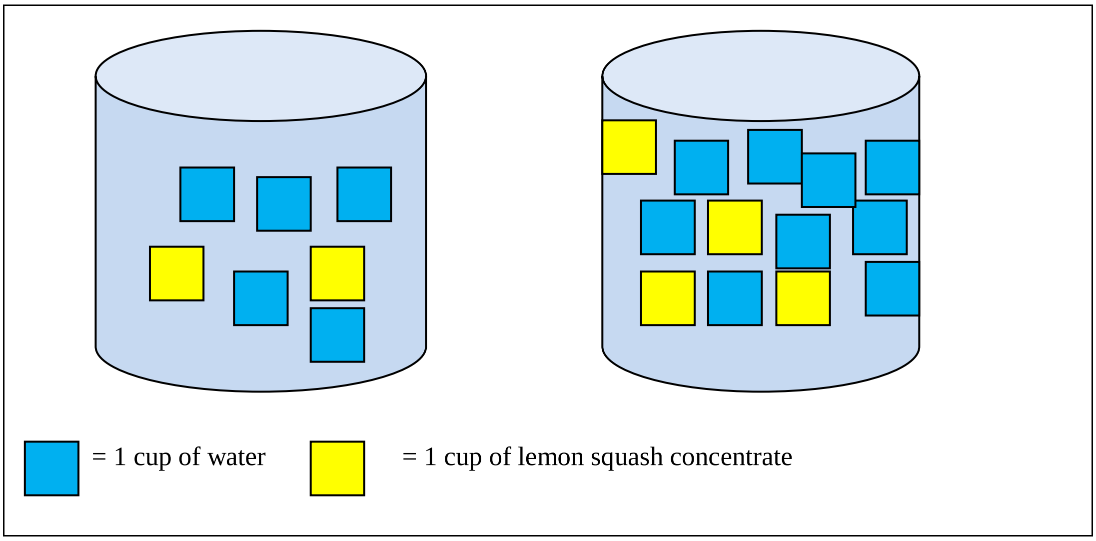
[33] Adapted from Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009) p.354, 355

Pupils must examine the cards and identify pairs of cards that illustrate equivalent ratios.

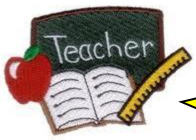


What equivalent ratios did you find?
Did you notice any patterns when you compared the ratios?
Can you identify any similarities between the way we work out equivalent fractions and equivalent ratios?

Lemon Squash Recipes



Show the pupils the above. They should work in pairs to attempt to solve the problem. Each container has the same amount of lemon squash. The squares represent the amount of water and squash in each container.



Which container will have the stronger lemon flavour or will they be the same? Give reasons for your answer.
Did you use fractions or ratios to find a solution?
How did you solve the problem?
How did you record your solution?

ICT Opportunities
[Link: Fish Ratio Exploration]

Reference List

Anghileri, j. (2007). *Developing number sense: Progression in the middle years*. New York: Continuum.

Askew, M. (2000). What does it mean to learn? What is effective teaching? In J. Anghileri (Ed.), *Principles and Practices in Arithmetic Teaching*. Buckingham Open University Press.

Askew, M. (2012). *Transforming primary mathematics*. Oxford: Routledge.

Boaler, J (2009), *The Elephant in the Classroom Helping Children Learn and Love Maths* Souvenir Press.

Cai, J, Sun, W (2002) Developing Students' Proportional Reasoning, A Chinese Perspective in *Making Sense of Fractions, Ratios and Proportion.* NCTM

Clarke, D, Roche, A, Mitchell, A (2008) 10 Practical Tips for Making Fractions Come Alive and Make Sense in Vol. 13, No.7 *Mathematics Teaching in the Middle School.*

Cramer, K, Wyberg, T, Leavitt, S (2008) The Role of Representations in Fraction Addition and Subtraction in Vol. 13, No.8 *Mathematics Teaching in the Middle School.*

Deboys, M, Pitt E, (1980), *Lines of Development in Primary Mathematics*, Queens University, Belfast

Fraivillg, J. L., Murphy, L. A. & Fuson, K. C. (1999). Advancing pupils' mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education*, *30*(2) 148-170.

Hufferd-Ackles, K., Fuson, K. C. & Gamoran Sherin, M. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, *35*(2) 81-116.

Lewis, C., Perry, R., Friedkin, S. & Baker, E. (2010). *Learning fractions in a linear measurement context: Development and fieldtests of a lesson study intervention*. Retrieved from http://www.lessonresearch.net/AERA%202010/AERA2010_MCLSG_FINAL.pdf

National Council for Curriculum and Assessment (1999). *Primary School Mathematics Curriculum*. Dublin: Stationery Office.

National Council for Curriculum and Assessment (1999). *Primary School Mathematics Curriculum: Teacher Guidelines*. Dublin: Stationery Office.

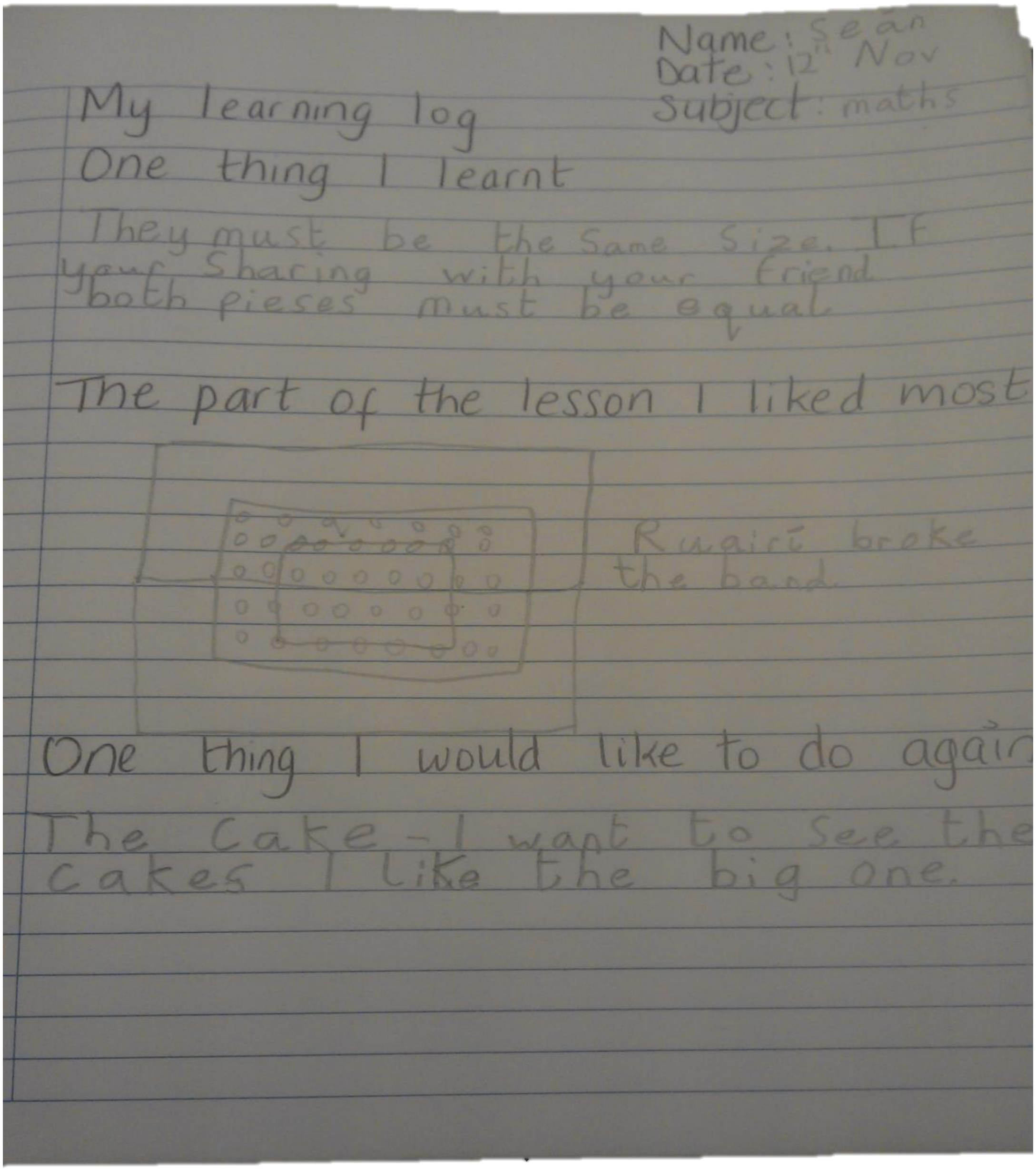
Suggate, J., Davis, A. & Goulding, M (2010). *Mathematical Knowledge for Primary Teachers*. 4th edn. Oxford: Routledge.

Van de Walle, J, Karp, Karen S, Bay-Williams, Jennifer M (2009). Elementary and Middle School Mathematics Teaching Developmentally.

Vygotsky, L.S. (1962). *Thought and Language*. Massachusetts: MIT Press.

Appendix A

Sample Learning Log



Name: Seán
Date: 12th Nov
Subject: maths

My learning log

One thing I learnt

They must be the same size. If your sharing with your friend both pieses must be equal.

The part of the lesson I liked most

Ruairí broke the band.

One thing I would like to do again

The cake - I want to see the cakes I like the big one.

Appendix B

Sample List of Questions and Teacher Language

Eliciting: Questioning and Teacher Language

Who did it another way?

Did anyone do it differently?

Did someone do it in a different way than Mary?

Is there another way of doing it?

John, could you share your method with all of us?

Mary has an interesting idea which I think would be useful for us to hear.

We will need to tidy our material away soon. Can we find a way of saving our ideas in another way?

Can anybody now explain to me in your own words what a _______ means?

Can anyone think of a way we can record this using numbers?

Using the material you have chosen to work with, can you show me what a _______ looks like?

How do you know?

How will we check?

Explain how you did it.

Supporting: Questions and Teacher Language

How about...?

Would it work if we...?

Could we...?

Harry suggests that...

So what you did was...

So you think that...

I have an idea...

Show me _________ in as many ways as possible.

Discuss with your partner what you need to do to solve the problem and estimate what the answer might be.

Mary has an interesting idea which I think would be useful for us to hear.

How could we draw a representation of this?

What materials could we use to represent this?

Extending: Questions and Teacher Language

Are there other ways of solving this?

Which is the most efficient way?

How could we do this in a more efficient way?

Which method is easiest to understand and why?

John, can you explain what Mary said in your own words?

Mary, can you repeat what John said using your own words?

Can you revoice what Mary said?

Check if this works for other examples.

How can we record the findings using symbols?

Record these using mathematical symbols.

Draw pictures to show how you worked out your answers.

Explain how you did this.

Explain how you got your answer.

Explain what we have discovered.

Write about what you have learned in your learning log.

What did you find difficult about solving this problem?

How did your answer compare to your estimate?

What patterns did you notice?

Did you spot any patterns?

Could you suggest an algorithm for working out an equivalent fraction?

Test your algorithm to see if it works.

Test your hypothesis using other problems.

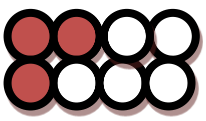
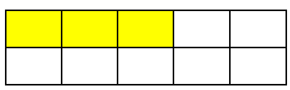
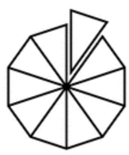
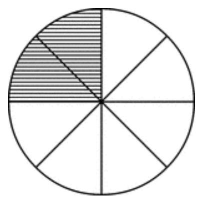
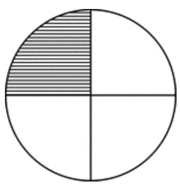
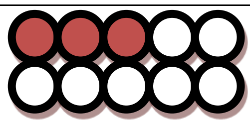
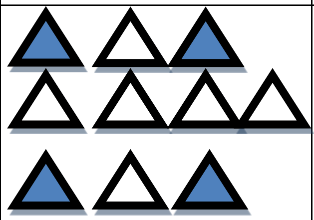
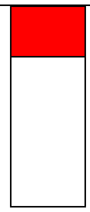
Appendix C

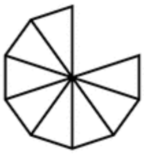
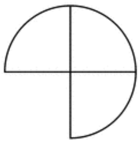
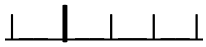
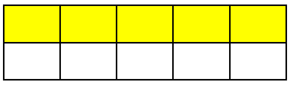
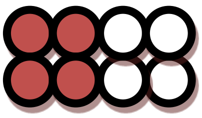
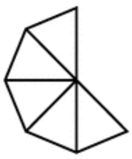
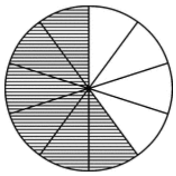
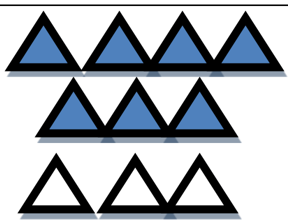
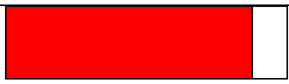
LESS THAN $\frac{1}{2}$	BETWEEN ½ AND 1	MORE THAN 1

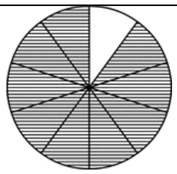
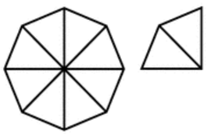
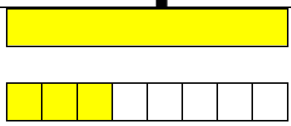
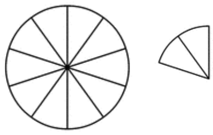
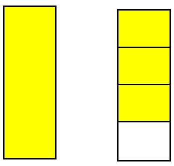
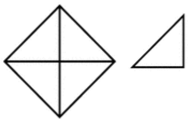
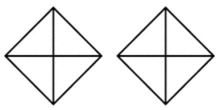
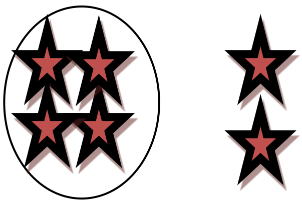
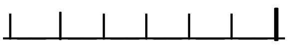
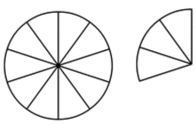
Fish for Fractions Benchmark Game

Challenge Cards

Fraction Cards

$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{2}{8}$
$\frac{3}{8}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$
 What fraction of the set is red?	 What fraction is yellow?	 What fraction is removed?	 What fraction is shaded?
 What fraction is shaded?	 What fraction of the set is red?	 What fraction of the set is blue?	 What fraction of the whole is shaded red?

 What fraction of the set is shaded blue?	 What fraction of the whole is missing?	 What fraction of the whole is missing?	 What fraction is marked on the number line?
 What fraction is marked on the number line?	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{5}{8}$
$\frac{6}{10}$	$\frac{6}{8}$	$\frac{7}{10}$	$\frac{7}{8}$
$\frac{8}{10}$	$\frac{9}{10}$	$\frac{2}{4}$	$\frac{3}{4}$
 What fraction is yellow?	 What fraction of the set is red?	 What fraction do you see?	 What fraction of the whole is shaded?
 What fraction is marked on the number line?	 What fraction of the set is blue?	 What fraction of the whole is shaded red?	 What fraction is marked on the number line?

 What fraction of the whole is shaded?	 What fraction of the set is shaded blue?	$\frac{10}{8}$	$\frac{11}{10}$
$\frac{11}{8}$	$\frac{12}{10}$	$\frac{12}{8}$	$\frac{9}{8}$
$\frac{7}{4}$	$\frac{6}{4}$	$\frac{5}{4}$	$\frac{8}{4}$
	 What fraction is shaded yellow?	 What fraction is marked on the number line?	
	 What fraction is shaded yellow?		
	 What fraction is marked on the number line?		

Clipart courtesy FCIT

Playing Instructions

This is a game for three players.

1. Cut out and laminate the 63 fractions cards. The 63 fraction cards are placed face down on the table. Each player takes a turn to select a card.

2. Each player selects a challenge card, either **Less than** $\frac{1}{2}$ **, Between** $\frac{1}{2}$ **and 1** or **More than 1.**

3. The aim of the game is to find and collect as many cards as possible that correspond to the individual player's challenge card. For example, if player A has the 'Less than $\frac{1}{2}$' game card and they turn over $\frac{3}{10}$, they identify that $\frac{3}{10}$ is less than $\frac{1}{2}$ and they collect that card. They can then take another turn. If this player turns over $\frac{6}{4}$, they replace the card, face down, and then player B takes their turn.

Play continues until:

a. A player has correctly collected a target number of cards, for example, 8 cards. They are declared the winner.

Or

b. All cards have been collected. The winner is the player with the most cards.

Or

c. The player with the most cards collected after a fixed period of time is declared the winner.

Appendix D

Counting in Fractions

First Class	$\frac{1}{2}$
Second Class	$\frac{1}{4}$
Third Class	$\frac{1}{8}$ $\frac{1}{10}$
Fourth Class	$\frac{1}{3}$ $\frac{1}{5}$ $\frac{1}{6}$ $\frac{1}{9}$ $\frac{1}{12}$

Counting activities should have:

- A lively pace
- Enthusiastic participation
- Two or three different short focussed activities (variety will maintain interest)
- Physical activity
- Choral response
- Individual response

There are many ways of counting in fractions which include:

Counting Stick

- Count in different fractions, for example, $\frac{1}{2}$, $\frac{1}{2}$. $\frac{1}{8}$
- Start at different starting points, count forwards and backwards
- Include mixed numbers and improper fractions, for example:
  - begin at one third, then count on by two third
  - begin at $1\frac{1}{2}$ and count on in $\frac{1}{2}$
- I need $1\frac{1}{2}$ metres of braid but my metre rules is in $\frac{1}{10}$ Count on the metre stick
- Name one end of the stick zero and the other end 10. Ask the pupils to estimate and give reasons for the position of $2\frac{1}{2}$, $6\frac{1}{2}$, 8, etc.

Stamp and Tap

Pupils find a space facing the board. Count forwards stamping feet in time. Stop at required number and turn in opposite direction. Now count back tapping their shoulders in time. (Do this without pausing!)

Human Number Line

Each pupil is given a large card with a fraction on it. Pupils are asked to line up from the smallest fraction to the largest. Teacher/pupil then discuss the order of the fractions, for example, before/ after, more than /less than/ same as, between, first/second, etc.

The Sound of a Number Game (Counting Can)

Teacher shows/tell the pupils the fraction of a unit being dropped into a tin. The pupils count silently in their heads as the teacher drops the fraction pieces into the tin. When the teacher stops, the pupil can call out the answer, or show its place on a number line. Teacher completes step one but this times ask pupils what fraction she/he would need to make 1 unit, 2 units etc. How many fraction pieces are in the tin, etc.

Feely bag Game

Place an even number of cubes in a bag, for example, 10. Tell the pupils you will give them half of the cubes in the bag. Count out five cubes. Ask the pupils how many cubes are still in the bag? How many cubes were in the bag at the beginning?

Stand and Sit Game

Pupils stand and then sit while saying the number sequence required, for example, Stand when our count is a whole unit. Pupils begin sitting and counting in quarters $\frac{1}{2}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ they stand and so on.

Count Around

Pupils stand in a circle and count around, each pupil saying the next number in the sequence. Start counting at $\frac{1}{2}$. The pupil who says number 2 sits down. Keep going until only one pupil is standing.

This could be differentiated in a number of ways including:

- using different fractions
- using different families of fractions
- using shorter/longer sequences
- using different starting/finishing points
- doing it backwards

Counting Choir

Divide class into 3 groups. Give each group a fraction out of the same family, for example, $\frac{1}{2}$ $\frac{1}{2}$.

Teacher plays the role of conductor with a baton. Teacher begins to count and then points the baton at one group to continue to count in unison. Teacher then points to a different group and continues.

Hand Game

Teacher picks a starting point, for example $\frac{3}{4}$. If teacher raises her/his hand up it means count $\frac{1}{2}$ more, if the hand faces down it means $\frac{1}{2}$ less.

A Fraction Wall

A fraction wall cut up into segments. Teacher hands out the segments to the pupils. The pupils put the fraction wall back together. Alternatively, a fraction wall with missing segments can be used. Pupils fill in the gaps on the fraction wall.

Guess my Number

I'm thinking of a fraction. It is between 0 and 1. The pupils then ask is it bigger than $\frac{1}{2}$? Is it smaller than $\frac{3}{4}$? Etc.

Target Boards

The Target Board is a very effective and versatile resource for mental / oral maths which can be placed on the whiteboard or wall. Each target board is a collection of numbers - in this case fractions. When using target boards encourage pupils to share their thinking and explain their mental methods. This helps pupils to realise there is more than one way to solve a problem. Explaining how you worked out something is a powerful way of learning. Examples of target board tasks for fractions include:

- Before/After, for example, what fraction comes 'after' X?
- Ordering from the lowest to the highest, for example, can you order the fractions on the second row?
- Estimation, for example, which column do you think has the greatest total?
- Count Forward
- Count Back
- Add the fractions in the first row

- Name two fractions that have the same denominator
- Name two equivalent fractions
- Name two/three fractions that are equivalent to one

Rope Activity

Stretch a skipping rope across the floor. Mark 0 at one end and 1 at the other end. Invite pupils to stand on or next to the rope to indicate positions of fractions, for example, $\frac{1}{2}$ a length of the rope or a $\frac{1}{2}$. Add an extra rope or two to extend the line to 2(3) so that the pupils can also represent improper fractions.

Comparing Halves:

Provide pupils with two different size wholes, each split into halves. Ask the pupils which half they would rather have? Discuss the difference between the halves and why one half is bigger than the other. Are they both halves? When can halves be different amounts?

The Frog and the Flea

A frog and a flea have a jumping competition. Each frog's jump was a $\frac{1}{3}$ of a unit long. Each flea's jump was $\frac{1}{2}$ of a unit long. The winner is the one who reaches four units in the fewest jumps. Predict who will win? Why? What if the distance was longer?

Number Line

1. Draw a number line on the floor. Mark units and half units/quarter units. Have the pupils jump in units, half units etc. counting as they go.
2. Pupils have a number line marked 0-1. Teacher calls out instructions, for example, put your counter on a fraction that comes just *before/after or on* any number *greater than/less than/ between*.
3. Empty number lines are also very useful, for example, in addition and subtraction of fractions pupils can represent the numbers to be added on the number line and this visual can help them with estimation.

Appendix E

Assessment

There are many forms of assessment which can be used effectively in mathematics lessons. The samples provided here are just a few. Please see the Assessment in the Primary School Curriculum: Guidelines for Schools (NCCA, 2007) for more information and guidance in relation to assessment - available at http://www.ncca.ie/uploadedfiles/publications/assess%20%20guide.pdf .

Two assessment checklists for fractions are provided here – one for whole class assessment and one for individual pupil assessment. The individual pupil assessment checklist can be used to 'track' a number of pupils in the class over the course of a year. Similarly, it can be used to 'track' pupils from 1st to 6th class. It enables a dual-approach to assessment – assessment of the concepts of fractions and also assessment of the developmental mathematical experiences (concrete, pictorial, abstract).

Assessment: Fractions Learning Trajectory Level A

Class Names: **Concepts**																															
Level A.1 **Find ½ of a set or shape**																															
Level A.2 **Find ¼ of a set or shape**																															
Level A.3 **Given ½ or a ¼ find a whole unit**																															

Class Assessment: Fractions Learning Trajectory Level B

Class Names: **Concepts**																															
Level B.1 **Compare,** **order,** **count and** **identify fractions and equivalent fractions with denominators 2,4,8,10**																															
Level B.2 **Calculate a unit fraction (with denominators 2,4,8,10) of a whole number**																															
Level B.3 **Calculate multiple fractions (with denominators 2,4,8,10) of a whole number**																															
Level B.4 **Calculate the number given the unit fraction (with denominators 2,4,8,10)**																															
Level B.5 **Calculate the number given the multiple fraction (with denominators 2,4,8,10)**																															

Level B.6 **Compare,** **order,** **count and** **identify fractions and** **equivalent fractions with** **denominators 3,5,6,9,12**																															
Level B.7 **Calculate a unit fraction** **(with denominators** **3,5,6,9,12)** **of a whole number**																															
Level B.8 **Calculate multiple fractions** **(with denominators** **3,5,6,9,12)** **of a whole number**																															
Level B.9 **Calculate the number given** **the unit fraction (with** **denominators 3,5,6,9,12)**																															
Level B.10 **Calculate the number given** **the multiple fraction (with** **denominators 3,5,6,9,12)**																															

Class Assessment: Fractions Learning Trajectory Level C

Class Names: **Concepts**																															
Level C.1 **Identify,** **construct,** **compare,** **order and** **count improper fractions**																															
Level C.2 **Express improper fractions as mixed numbers and vice versa**																															
Level C.3 **Construct algorithm for equivalent and simplified fractions**																															

Class Assessment: Fractions Learning Trajectory Level D

Class Names: **Concepts**																															
Level D.1 **Add and subtract fractions (first with like denominators, then repeat with unlike denominators)**																															
Level D.2 **Add and subtract mixed numbers**																															
Level D.3 **Multiply a fraction by a whole number**																															
Level D.4 **Multiply a fraction by a fraction**																															
Level D.5 **Divide a whole number by a unit fraction**																															

Class Assessment: Fractions Learning Trajectory Level E

Class Names: **Concepts**																															
Level E.1 **Understand and use simple ratios**																															

Individual Pupil Assessment: Fractions Learning Trajectory

Pupil's Name:	**Developmental Experiences**		
Concepts	**Concrete**	**Pictorial**	**Abstract**
Level A.1 **Find ½ of a set or shape**			
Level A.2 **Find ¼ of a set or shape**			
Level A.3 **Given ½ or a ¼ find a whole unit**			

Pupil's Name:	Developmental Experiences		
Concepts	Concrete	Pictorial	Abstract
Level B.1 Compare, order, count and identify fractions and equivalent fractions with denominators 2,4,8,10			
Level B.2 Calculate a unit fraction (with denominators 2,4,8,10) of a whole number			
Level B.3 Calculate multiple fractions (with denominators 2,4,8,10) of a whole number			
Level B.4 Calculate the number given the unit fraction (with denominators 2,4,8,10)			
Level B.5 Calculate the number given the multiple fraction (with denominators 2,4,8,10)			

	Concrete	Pictorial	Abstract
Level B.6 **Compare,** **order,** **count and** **identify fractions and** **equivalent fractions with** **denominators 3,5,6,9,12**			
Level B.7 **Calculate a unit fraction** **(with denominators** **3,5,6,9,12)** **of a whole number**			
Level B.8 **Calculate multiple fractions** **(with denominators** **3,5,6,9,12)** **of a whole number**			
Level B.9 **Calculate the number given** **the unit fraction (with** **denominators 3,5,6,9,12)**			
Level B.10 **Calculate the number given** **the multiple fraction (with** **denominators 3,5,6,9,12)**			

Pupil's Name:	Developmental Experiences		
Concepts	**Concrete**	**Pictorial**	**Abstract**
Level C.1 **Identify, construct, compare, order and count improper fractions**			
Level C.2 **Express improper fractions as mixed numbers and vice versa**			
Level C.3 **Construct algorithm for equivalent and simplified fractions**			

Pupil's Name:	Developmental Experiences		
Concepts	Concrete	Pictorial	Abstract
Level D.1 **Add and subtract fractions (first with like denominators, then repeat with unlike denominators)**			
Level D.2 **Add and subtract mixed numbers**			
Level D.3 **Multiply a fraction by a whole number**			
Level D.4 **Multiply a fraction by a fraction**			
Level D.5 **Divide a whole number by a unit fraction**			

Pupil's Name:	Developmental Experiences		
Concepts	Concrete	Pictorial	Abstract
Level E.1 **Understand and use simple ratios**			

Appendix F

Teacher Reflection

Placing yourself on a continuum is a simple way for teachers to reflect on their own practice. The following reflection continuum (Table G.1) is adapted from work completed by Hufferd-Ackels, Fuson & Sherin (2004) when they devised a developmental learning trajectory based on classroom mathematical discourse and thinking.

Table G.1 Reflection Continuum for Teachers

Questioning	Explaining Mathematical Thinking	Source of Mathematical Ideas	Responsibility for Learning
Shift from teacher as questioner to student and teacher as questioners	Pupils increasingly explain and articulate their mathematical ideas to the teacher and to others	Shift from teacher as the source of ideas to pupils' ideas also influencing the direction of lesson	Pupils increasingly take responsibility for learning and evaluation of others and self. Maths sense becomes the criterion for evaluation
Highlight or circle the most appropriate word in each section. Include the date and revisit after a particular instructional period.			
Always Sometimes Never	Always Sometimes Never	Always Sometimes Never	Always Sometimes Never

Appendix G

Sample Digit Cards

$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$
$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$
$\frac{7}{4}$	$\frac{8}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$\frac{5}{5}$	$\frac{6}{5}$	$\frac{10}{5}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$
$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\frac{12}{6}$
$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$
$\frac{7}{8}$	$\frac{8}{8}$	$\frac{9}{8}$	$\frac{10}{8}$	$\frac{16}{8}$	$\frac{32}{8}$
$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{6}{9}$
$\frac{7}{9}$	$\frac{8}{9}$	$\frac{9}{9}$	$\frac{10}{9}$	$\frac{11}{9}$	$\frac{18}{9}$
$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$
$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$	$\frac{11}{10}$	$\frac{20}{10}$

Appendix H

Fraction Dice Game

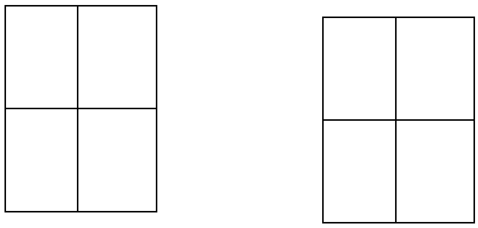
Playing Instructions**:**

Version 1:

This version uses the templates for halves and quarters fractions only. Print out and laminate two copies of the halves (in blue) and two copies of the quarters (in red). Magnetise the pieces if you intend playing the game in a whole-class setting on the magnetic whiteboard.

Label a die $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$.

Draw two grids on the whiteboard 32cm x 32cm. There are four 16cm x 16 cm squares inside this grid.



One grid is team A's game board and the other is team B's game board. In preparation for the game, ask children to select pieces to create, for example, $\frac{3}{4}$ in as many ways as possible:

- $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ (using the square pieces)
- $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ (using triangular pieces)
- $\frac{1}{2} + \frac{1}{4}$ (using square pieces)
- $\frac{1}{2} + \frac{1}{4}$ (using triangular pieces)

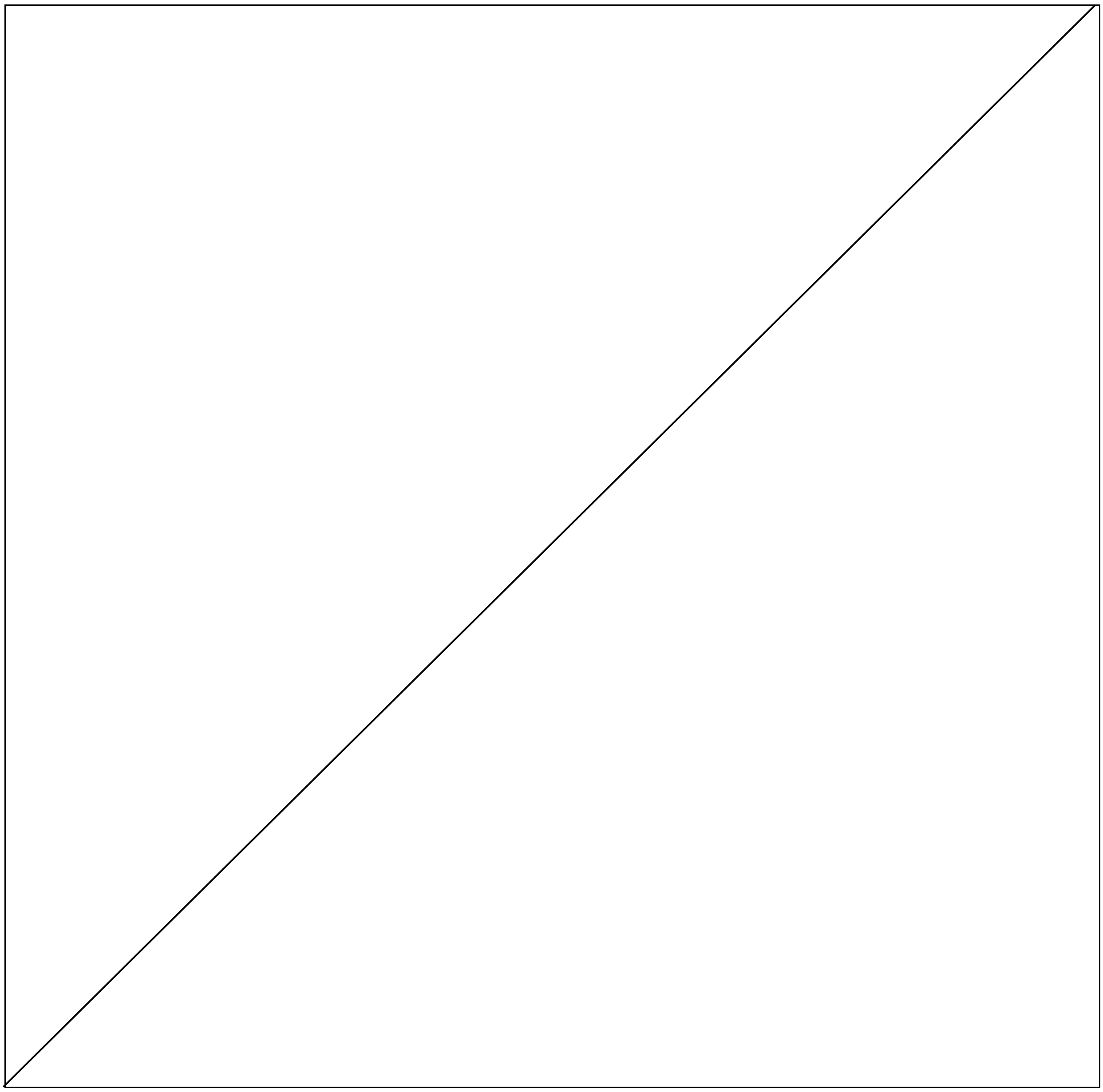
Place all the pieces onto the grid before playing. Teams take turns throwing the die and they remove the corresponding fraction piece from the board. The aim is to be the first to team to empty their board completely. If a piece cannot be removed, the team misses a turn. The game can also be played in reverse, where the teams fill the empty grid with fraction pieces. Consider also playing the game with pairs of pupils.

Version 2:

This version makes use of the eighths fraction pieces. Play as above, but add two more square grids to your game board. Label your die $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{4}, \frac{3}{8}, \frac{5}{8}$

Halves

1. print on blue card/blue paper
2. laminate
3. cut out each half

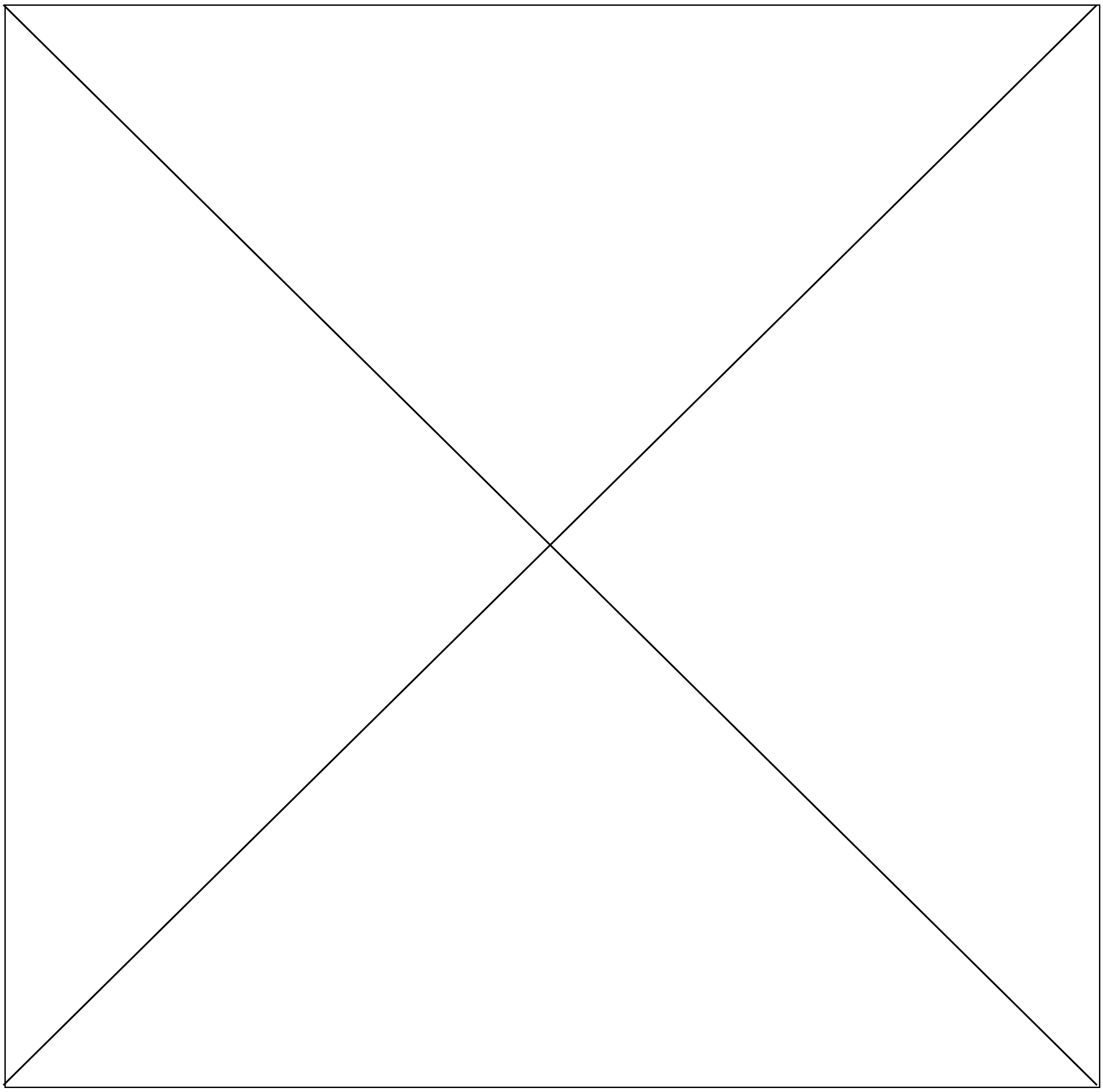


Halves

1. print on blue card/blue paper
2. laminate
3. cut out each half

Quarters

1. print on red card/red paper
2. laminate
3. cut out each quarter



Quarters

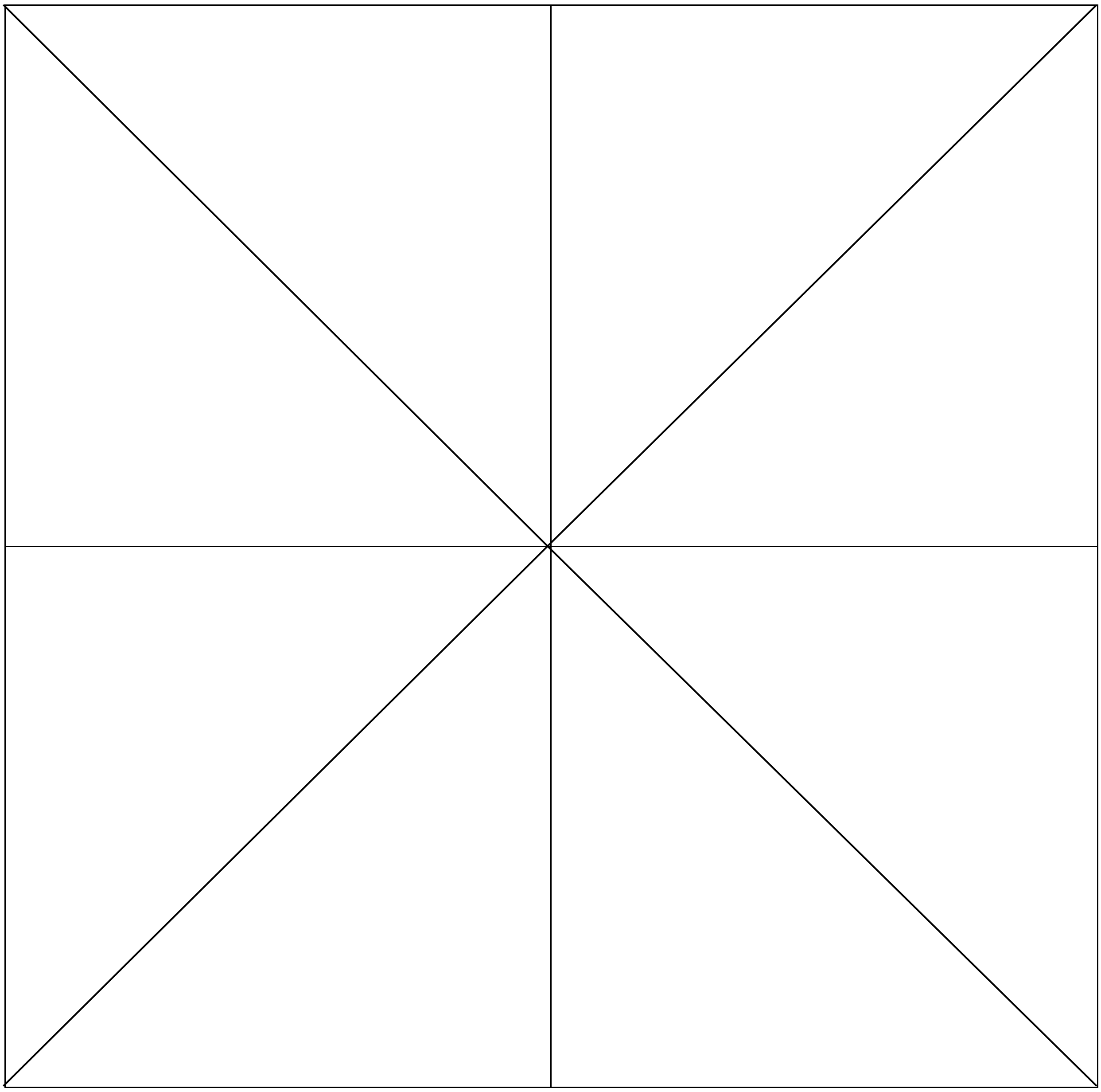
1. print on red card/red paper
2. laminate
3. cut out each quarter

Version 2:

Eighths

To play the second version of the game these eighths are required:

1. print on yellow card/yellow paper
2. laminate
3. cut out each eighth



Eighths

To play the second version of the game these eighths are required:

1. print on yellow card/yellow paper
2. laminate
3. cut out each eighth