# 4.4.5 - De Moivre's Theorem II - Proof

4.4 - Algebra - Complex Numbers

Leaving Certificate Mathematics

Higher Level ONLY

# De Moivre's Theorem

**Theorem:** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$ for **all** $n$.

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**Step 1:** Test for $n = 1$.

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$$\text{LHS} \quad = \quad (\cos\theta + i\sin\theta)^1$$

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$$\begin{aligned} \text{LHS} &= (\cos\theta + i\sin\theta)^1 \\ &= \cos\theta + i\sin\theta \end{aligned}$$

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Proof: Use induction.

**Step 1:** Test for $n = 1$.

$$\begin{aligned} \text{LHS} &= (\cos\theta + i\sin\theta)^1 \\ &= \cos\theta + i\sin\theta \end{aligned} \qquad \text{RHS} = \cos(1\theta) + i\sin(1\theta)$$

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LHS = RHS

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LHS = RHS

$\therefore$ Equation is true for $n = 1$.

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$$(\cos\theta + i\sin\theta)^{k+1}$$

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$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$

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$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \qquad \ldots \text{assumed}\end{aligned}$$

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$$\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \qquad \ldots\text{assumed} \\
&= \cos(k\theta + \theta) + i\sin(k\theta + \theta)
\end{aligned}$$

**Theorem:** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{for } n \in \mathbb{N}.$

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&= \cos(k\theta + \theta) + i\sin(k\theta + \theta) \\
&= \cos\left[(k+1)\theta\right] + i\sin\left[(k+1)\theta\right]
\end{aligned}$$

**Theorem:** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{for } n \in \mathbb{N}.$

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**Step 4: Conclusion.**

**Theorem:** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{for } n \in \mathbb{N}.$

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- True for $n = k + 1$,

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- True for $n = k + 1$, assuming true for $n = k$.
- But true for $n = 1$.

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**Step 4: Conclusion.**

- True for $n = k + 1$, assuming true for $n = k$.
- But true for $n = 1$.
- $\therefore$ True for $n = 2$,

**Theorem:** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{for } n \in \mathbb{N}.$

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- True for $n = k+1$, assuming true for $n = k$.
- But true for $n = 1$.
- $\therefore$ True for $n = 2, 3,$

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**Step 4: Conclusion.**

- True for $n = k+1$, assuming true for $n = k$.
- But true for $n = 1$.
- $\therefore$ True for $n = 2, 3, \ldots$ and all $n \in \mathbb{N}$

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**Step 4: Conclusion.**

- True for $n = k+1$, assuming true for $n = k$.
- But true for $n = 1$.
- $\therefore$ True for $n = 2, 3, \ldots$ and all $n \in \mathbb{N}$ **by induction**.