Teaching & Learning Plans

Complex Number Operations

Leaving Certificate Syllabus

Project Maths
Tionscadal Mata
Development Team

The Teaching & Learning Plans are structured as follows:

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Aims outline what the lesson, or series of lessons, hopes to achieve.

Prior Knowledge points to relevant knowledge students may already have and also to knowledge which may be necessary in order to support them in accessing this new topic.

Learning Outcomes outline what a student will be able to do, know and understand having completed the topic.

Relationship to Syllabus refers to the relevant section of either the Junior and/or Leaving Certificate Syllabus.

Resources Required lists the resources which will be needed in the teaching and learning of a particular topic.

Introducing the topic (in some plans only) outlines an approach to introducing the topic.

Lesson Interaction is set out under four sub-headings:

i. **Student Learning Tasks – Teacher Input:** This section focuses on teacher input and gives details of the key student tasks and teacher questions which move the lesson forward.

ii. **Student Activities – Possible and Expected Responses:** Gives details of possible student reactions and responses and possible misconceptions students may have.

iii. **Teacher's Support and Actions:** Gives details of teacher actions designed to support and scaffold student learning.

iv. **Checking Understanding:** Suggests questions a teacher might ask to evaluate whether the goals/learning outcomes are being/have been achieved. This evaluation will inform and direct the teaching and learning activities of the next class(es).

Student Activities linked to the lesson(s) are provided at the end of each plan.

Teaching & Learning Plans: Complex Number Operations

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Aims

To familiarise students with operations on Complex Numbers and to give an algebraic and geometric interpretation to these operations

Prior Knowledge

- The Real number system and operations within this system
- Solving linear equations
- Solving quadratic equations with real and imaginary roots
- Translations, reflections, rotations
- Rules for indices and surds for example $\sqrt{xy} = \sqrt{x}.\sqrt{y}$ provided both are not negative
- Powers of i
- Modulus of a Complex Number

Learning Outcomes

As a result of studying this topic, students will be able to

- add and subtract Complex Numbers and to appreciate that the addition of a Complex Number to another Complex Number corresponds to a translation in the plane
- multiply Complex Numbers and show that multiplication of a Complex Number by another Complex Number corresponds to a rotation and a scaling of the Complex Number
- find the conjugate of a Complex Number
- divide two Complex Numbers and understand the connection between division and multiplication of Complex Numbers

Catering for Learner Diversity

In class, the needs of all students whatever their ability level are equally important. In daily classroom teaching, teachers can cater for different abilities by providing students with different activities and assignments graded according to levels of difficulty so that students can work on exercises that match their progress in learning. Some students may only be able to engage in activities which are relatively straightforward, while others may be able to engage in more open-ended and challenging activities. Selecting and assigning activities appropriate to a student's ability will cultivate and sustain his/her interest in learning.

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In interacting with the whole class, teachers can employ effective and inclusive questioning. Questions can be pitched at different levels and can move from basic questioning to ones which are of a higher order nature. In this T & L Plan, some students may be required to answer a question such as: **What is the distance from the origin to i?** A more challenging question can be reserved for others: **Predict what will happen to Complex Numbers when you subtract a Complex Number, z from each one.**

Besides whole-class teaching, teachers can consider different grouping strategies – such as group and pair work – to encourage student interaction, help students to verbalise their mathematical understanding and help to build student self-confidence and mathematical understanding. For example in this T & L Plan students are asked to share in pairs what connections they see when they add the same Complex Number to other Complex Numbers.

Relationship to Leaving Certificate Syllabus

Sub-Topic	**Learning outcomes**		
Students learn about	**Students working at FL should be able to**	**In addition, students working at OL should be able to**	**In addition, students working at HL should be able to**
3.1 Number Systems		• Illustrate Complex Numbers on an Argand diagram • investigate the operations of addition, multiplication, subtraction and division with Complex Numbers in the form $a + ib$ • interpret the Modulus as distance from the origin on an Argand Diagram and calculate the complex conjugate	• Calculate conjugates of sums and products of Complex Numbers

Resources Required

Graph paper, geometry instruments, coloured pencils, Antz clip downloaded from YouTube

Introducing the Topic – (Appendix 1 and 2, page 26-28)

A rationale for Complex Numbers needs to be established to show students why the real number system needs to be extended to deal with getting the square roots of negative numbers and to showcase the unique two dimensional nature of this number system. One such approach would be to use simple equations outlined in Appendix 1, page 26-27 to see the need for the different number systems. This approach reflects the development of the topic throughout our mathematical history. It also starts from an exploration by the student, leading to a discovery. This could be supplemented and/or reinforced by the students accessing John and Betty's Journey into Complex Numbers from the following website: **http://mathforum.org/johnandbetty/frame.htm.** For an historical account of Complex Numbers refer to Appendix 2, page 28.

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Real Life Context

Complex Numbers are useful in representing a phenomenon that has two parts varying at the same time, for example an alternating current. Also, radio waves, sound waves and microwaves have to travel through different media to get to their final destination. There are many instances where, for example, engineers, doctors, scientists, vehicle designers and others who use electromagnetic signals need to know how strong a signal is when it reaches its destination. The two parts in this context are: the rotation of the signal and its strength. The following are examples of this phenomenon:

- A microphone signal passing through an amplifier
- A mobile phone signal travelling from the mast to a phone a couple of miles away
- A sound wave passing through the bones in the ear
- An ultrasound signal reflected from a foetus in the womb
- The song of a whale passing through miles of ocean water

Complex Numbers are also used in:

- The prediction of eclipses
- Computer game design
- Computer generated images in the film industry
- The resonance of structures (bridges, etc.)
- Analysing the flow of air around the wings of a plane in aircraft design

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
Section A: Revision and assessment of Number Systems, quadratic equations, powers of i and modulus of a complex number			
Note: Students should complete **Section A, Student Activity 1-5, page 30-38** to reinforce their existing knowledge base of Number Systems, quadratic equations, powers of i and modulus of a Complex Number.	» Students complete **Section A, Student Activity 1-5**	» Distribute **Section A, Student Activity 1-5**	
» Put a tick in the box that best describes what you can do.	» Students put a tick in the box that best describes what they can do.	» Distribute **Section A, Student Activity 6, page 39.**	» Assess from **Section A, Student Activity 6** where students' learning is at before moving onto Section B.

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Lesson Interaction			
Student Learning Tasks: Teacher Input	**Student Activities: Possible and Expected Responses**	**Teacher's Support and Actions**	**Checking Understanding**
Section B: The addition of Complex Numbers - Addition of a Complex Number seen as a translation			
» If we are to consider Complex Numbers as a number system, what was the first thing we learned to do with every other number system? » In this lesson we are going to explore addition, subtraction, multiplication and division of Complex Numbers and discover what happens when you apply these operations using algebra and geometry.	• Add, subtract, multiply and divide	• Prepare the Board Plan **(Appendix 3, page 29)**.	
» Record what you will be learning in class today.	» Students write what they will be learning in class today.		» Have students managed to write in their own words the learning outcome?

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» We will now look at a video clip of an animation film (Antz clip from YouTube) and over the course of the lesson we will discover what operations of Complex Numbers could have been used to allow the creators of this animation to move objects around the screen.		» Play clip from Antz video	
» Let us start with addition. » How would we go about adding two Complex Numbers for example: $(4 + i) + (2 + 2i)$, $(3 - 5i) + (5 + 2i)$, $(6 + i) + (4 - 3i)$?	» Student should try out a few examples, compare answers around the class and have a discussion about why the answers are not all agreeing.	» Put the examples $(4 + i) + (2 + 2i)$, $(3 - 5i) + (5 + 2i)$ and $(6 + i) + (4 - 3i)$ on the board. » Give students time to explore and investigate for themselves.	
» How did you go about $(4 + i) + (2 + 2i)$?	• Add the 4 to the 2 and the i to the $2i$		
» What do you think is the rule for addition? » Does this make sense?	• Add the real parts and add the imaginary parts separately. • Yes, because it's like adding like terms in Algebra. (Students are displaying their previous learning from Algebra here).	» Add the word "**real**" and "**imaginary**" to the word bank. (Appendix 3)	

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» (Dealing with students who are demonstrating poor procedural knowledge): Why couldn't $8i$ be an answer to this question? Talk me through where I went wrong.	• You can't add $4 + i$ together because they are in different dimensions.	» Write the incorrect solution to the above example on the board: $4 + i = 4i$ and $2 + 2i = 4i$ $4i + 4i = 8i$. » Following the discussion, erase	**Note:** If some students are having difficulty a more scaffolded approach could be used where the students begin by adding imaginary components first such as: 'what do I get when I add i to $2i$'? Then introduce addition of the real parts.
» Now with your partner come up with a wording for "Addition of Complex Numbers Rule #1"and when you are happy with it write it in the box provided in your rule card. **Note:** Encourage HIGHER LEVEL students to write the rule in the general case). » Follow this with 2 examples of your own to illustrate this.	» Students may write: • Add the real parts and the imaginary parts • For Complex Numbers $a + bi$ and $c + di$, the rule is: $(a + c) + i(b + d)$	» Distribute **Section B, Student Activity 1 page 40.** **Note:** Students are more likely to learn with understanding if they have tried to extend their existing knowledge rather than be prescribed a 'rule' to follow from the start.	» In conversation with students are they using words and phrases such as 'like terms' and 'real' and 'imaginary'?
		» Write **"The Addition of Complex Numbers"** on the board **Re + Re and Im + Im** $(a + bi) + (c + di)$ $(a+c) + i\,(b+d)$ **(Appendix 3, page, 29).**	

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» In pairs, we are now going to investigate what happens to Complex Numbers when we add the same Complex Number to them. » When you are finished swap and discuss.	» Students complete **Section B, Student Activity 2, page 41.** » Students swap their work and discuss.	» Distribute **Section B, Student Activity 2, page 41.** » Draw the Argand Diagram on the board. GeoGebra or similar software can also be used in the exploration of this task. » Circulate to monitor student progress.	» Students are checking each others work. » Are students comfortable applying Rule #1? » Are students confident in plotting points accurately on an Argand Diagram? » If students appear to have difficulty allow them to talk through their work so that they can identify the area of weakness.
» What do you notice about the lines you've drawn on the Argand Diagram? **Note:** Care is needed so that students understand the difference between lines on the Cartesian Plane and lines on an Argand Diagram.	• The lines are parallel. • The lines are the same length. • They're going in the same direction. • When you join the points you could form a parallelogram.		
» Share with your partner what connections you see when you add the same Complex Number to other Complex Numbers? Then record your findings.	• When we add $4 + i$ to different Complex Numbers each one is the same distance and in the same direction. • It's a translation.	» Write the word **"translation"** in the word bank and then circulate and check that students know what a translation means.	» Do they see that every point in the plane is moved the same distance and in the same direction when a particular Complex Number z is added to it? » Some students at this stage may make a reference to the Antz clip.

Student Learning Tasks Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
Section C: The Subtraction of Complex Numbers			
» Now let's see what happens when we subtract Complex Numbers. » How would we go about subtracting two Complex Numbers for example: $(2 + 2i) - (4 + i)$, $(2 - 5i) - (1 + 2i)$, $(2 - i) - (4 - 3i)$?	» Student should try out a few examples, compare answers around the class and have a discussion about why the answers are not all agreeing.	» Give students time to explore possibilities and to discuss what is happening.	» Are students applying knowledge of addition of Complex Numbers? » Are students extending their existing knowledge of subtracting negative quantities and the importance of using brackets correctly?
» What do you think the rule for subtraction is?	• It's like the addition rule except we're subtracting $(2 + 2i) - (4 + i)$ $= 2 + 2i - 4 - i = -2 + i$ • It's like in algebra when you have two terms. • You're subtracting the whole expression not just the first term. • When you put a minus in front of the first term you might think you're only subtracting the first term of the expression. You must use a bracket to ensure you don't forget to subtract all of the terms in the expression.	» Write $w_1 - z$ $= (2 + 2i) - (4 + i)$ on the board.	
		» Write: "**brackets**" and "**subtract all terms**" in the word bank, (Appendix 3, page 29).	

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
Note: If students are making procedural errors such as $(2 + 2i) - (4 + i) = -2 + 3i$, you could ask this question: What's wrong with this solution? Talk me through where you think you could have gone wrong.		» Engage students in talking about common mathematical errors.	
» Now write the rule for "Subtraction of Complex Numbers, Rule # 2" in **Section B, Student Activity 1**. **Note:** Encourage HIGHER LEVEL students to represent the rule in the general case. » Follow this with two examples of your own to illustrate this.	» Students may write: • Re - Re and Im – Im • $(a + bi) - (c + di) = (a - c) + i(b - d)$ » Students check each others work	» Write the subtraction rule on the board: Re - Re and Im – Im $(a + bi) - (c + di) = (a - c) + i(b - d)$ (Appendix 3, page 29) » Circulate to check on students' progress.	» Are students' examples showing their understanding of the rule?
» Go back to **Section B, Student Activity 2** and predict what will happen to the Complex Numbers if you subtract z from each one?	• They will go in the opposite direction	» Circulate to monitor students' progress.	

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» Now do the calculation and represent the results on the same Argand Diagram as before using a different colour pencil. » What do you notice? » What direction and distance has it moved?	• This is a translation in the opposite direction to $w_1 z$ i.e. translation $-oz = zo$. • The same distance but in the opposite direction	» Draw Argand Diagram. (Alternatively, this could be explored using a computer package such as GeoGebra.) » Write in the word bank: **"translation in opposite direction" (Appendix 3, page 29)**	» Do students see that adding a Complex Number z to Complex Number w is a translation oz on w, whereas subtracting z is a translation zo on w?
» Complete **Section C, Student Activity 1.**	» Students should try **Section C, Student Activity 1,** compare answers around the class and have a discussion about why the answers are not all agreeing.	» Distribute **Section C, Student Activity 1, page 42.**	

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
Section D: The Multiplication by a Complex Number whose imaginary part = 0 i.e. by a Real number			
» Now let's look at the multiplication of Complex Numbers. You are going to investigate what happens when you multiply Complex Numbers through a series of practical activities. Let's begin with **Section D, Student Activity 1** which examines what happens when we multiply a Complex Number by a real number.		» Distribute Section **D, Student Activity 1, page 43.** » Give students time to investigate for themselves. » Check that students are plotting the points on the Argand Diagram.	
» Let's compare our answers around the class and have a discussion about why the answers are not all agreeing.	» The **Student Activity** should enable students to offer some of these responses: Q1 • $2(3 + 4i) = 6 + 8i$. Q3 • I can use a ruler or I can calculate the distance using the modulus or use Pythagoras' Theorem. Q4 • $2z$ is twice as far from the origin as z. • The distance has doubled. • All the points are on the same straight line. • The number we are multiplying by is a positive whole number greater than 1.		» Do students see that multiplication by a real number corresponds to a scaling - multiplying by a real number >1 "expands". » Do students recall how to get the modulus of a Complex Number and what it means?

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» Another way of describing what happens when you multiply by any real number greater than 1 is: it expands or stretches or scales, it is enlarged. It increases the modulus or distance of z from the origin on the same straight line by a factor equal to the real number. » Turn to your partner and using mathematical language describe the multiplication of $3 + 4i$ by a real number greater than one.	• When we multiplied $3 + 4i$ by 5 the modulus of $3 + 4i$ was stretched/expanded or enlarged on the same straight line by a factor of 5.	» Write in the word bank: **scales, enlarged, stretches, expands, factor**	» Are students using the mathematical terms in a suitable context?
» If we say that multiplying by 2 "stretches" or 'scales' or 'expands' or enlarges the modulus of a Complex Number, what happens when we multiply by a number between 0 and 1? » Another way of describing this is, multiplying by a real number between 0 and 1 "contracts" the line joining the Complex Number to the origin. » What would happen if you multiply by a negative number? Predict what would happen?	• The distance is shorter. • The modulus is less. • Would it go in the other direction?	» Write in word bank: **contracts** » Give students time to think.	
» Calculate and plot on the Argand Diagram $-z$, $-2z$, $-z/2$, $-1.5z$. Investigate and write about what happens. **Note:** $z = 3 + 4i$	**Note:** Students should adopt an investigative approach. This will contribute significantly to better understanding. » Students should compare their answers around the class and have a discussion about what happens.	» Write $-z$, $-2z$, $-z/2$, $-1.5z$ on the board.	

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
Section E: The Multiplication of a Complex Number by another Complex Number whose real part =0 i.e. a purely imaginary number			
» Now we will look at multiplication by a purely imaginary number. Examine the effect of multiplying $5 + 3i$ by i algebraically? » Now come up with two other multiplication by i questions, do out the solutions and pass the questions to your partner to solve.	» Students work on the problem $i(5 + 3i)$ $= 5i + 3i^2$ $= 5i - 3$ $= -3 + 5i$ » Students compose questions, solutions. » Students swap questions and do out solutions to their partners' questions. » Students compare answers.	» Write i**(5 + 3**i**)** on the board.	» Can students apply the rules of indices in the multiplication of i by i? » Can they recall that $i^2 = -1$
» Let's explore multiplication by i geometrically: Using the Argand Diagram explain to your partner how the results 1, i, -1 and $-i$ were achieved.	• When you multiply 1 by i you get i. • When you multiply i by i you get -1. • When you multiply -1 by -i you get $-i$. • When you multiply i by $-i$ you get 1. • We know that if i is raised to any power the result will be either i, -1, -i or 1.	» Write **{-1, 1,** i**, -i}** on the board. » Draw the Argand Diagram, label the axes and plot the four elements of the set.	

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» What is the distance from the origin to 1? » What is the distance from the origin to i? » What is the angle formed by the arms joining the origin to 1 and i? » Is it true for i and -1, -1 and $-i$ and $-i$ and 1? » Can you describe what's happening here? Does it you remind you of anything?	• 1 • 1 • 90° • Right angle • Yes because we're multiplying by i each time. • It's like a clock but it's moving in the opposite direction. • It's anti-clockwise. • It makes a circle. • The point is moving around. • The point is turning 90° each time.		
» We call this a rotation of 90 degrees.		» Use a compass on the board or Geogebra to demonstrate this rotation. » Write **rotation** in the word bank	
» Now complete **Section E, Student Activity 1.**		» Distribute **Section E, Student Activity 1, page 44.**	

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» Now we will look again at the Antz video. While you are watching it, can you speculate on a possible connection between the movement of the Ant General and the Army of Ants and the operations of Complex Numbers we have encountered so far? Share your ideas with your partner.	• The Ant Army are moving in straight lines in the same direction. • Addition of Complex Numbers is used to move points and objects around the screen in straight lines • The general is moving and turning – this is like multiplication of Complex Numbers	» Play Antz video	» Do students see the real life context in this animation?
» We saw already that multiplication by i causes a rotation of 90°. If we want to rotate by different angles we multiply by Complex Numbers » Let's look at an example such as $(3 + i)(1 + 2i)$ that combines real and imaginary numbers; we will use previous work to examine it algebraically first. How do we do this?	• $(3 + i)(1 + 2i)$ $= 3(1 + 2i) + i(1 + 2i)$ $= 3 + 6i + i + 2i^2$ $= 1 + 7i$	» Write $\mathbf{(3 + i)(1 + 2i)}$ on the board	» Are students using their prior knowledge to multiply two binomials using the distributive law?
» Record rule for "Rule # 3 Multiplication of Complex Numbers" in **Section B; Student Activity 1** **Note:** Encourage Higher Level students to represent it in the general case. » Illustrate with two examples of your own.		» Write on board: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$	» Are students simplifying i^2 to -1?

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» You will now investigate what happens geometrically when two Complex Numbers are multiplied.	» Students explore and investigate **Section E, Student Activity 2.**	» Distribute **Section E, Student Activity 2, page 45.**	» Are students using protractors or are they using Trigonometry? » Can students see the connection between the modulus? » Are students making the connection about the angles?
	» Student should compare answers around the class especially to parts (4) and (6) and have a discussion about the connections they made.	» Allow students time to discuss their answers.	
» What do you notice about the angles? » What do you notice about the moduli?	• The angles for $3 + i$ and $1 + 2i$ add to give the angle for $1 + 7i$. • The angle between the Real axis and $3 + i$ is equal to the angle made between $1 + 2i$ and $1 + 7i$. • When you multiply the moduli of the Complex Numbers you get the modulus of $1 + 7i$, which is the answer.		

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» Summarise what you have learned from **Section E, Student Activity 2.**	• Students present their summaries to the class.	» Circulate and read students' summaries. Ask several students if they will read their summary to the class and explain what they have discovered in **Section E, Student Activity 2.**	» Can students verbalise what they have discovered?
» For HIGHER LEVEL students: the names of these angles are the argument of $(1 + 2i)$, arg $(1 + 2i)$ for short, arg $(3 + i)$ and arg $(1 + 7i)$. **Note:** Even though polar coordinates are not introduced here, it might be no harm for HIGHER LEVEL students to realise that a Complex Number could just as well be given in terms of its modulus and argument. They can be told that this is something they are likely to meet later (in strand 4; De Moivre's theorem).		» Write on the board: **Add the Angles (Arguments) Multiply the Moduli**	
» Complete **Section E, Student Activity 3**.		» Distribute **Section E, Student Activity 3, page 46-47**.	

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
Section F: The Complex Conjugate			
» What happens when we multiply out these Complex Numbers? $(5 + 2i)(5 - 2i)$ $(3 - 7i)(3 + 7i)$	» Students multiply out these complex numbers. • The answer is a REAL number • The imaginary parts cancel out	» Write (5 + 2i)(5 – 2i) on the board.	» Are students applying the multiplication rule and performing the algebraic calculations with precision? Are they using the Complex Number terms 'real' and 'imaginary'?
» What do you notice about these numbers? i.e. $(5 + 2i)$ and $(5 - 2i)$ $(3 - 7i)$ and $(3 + 7i)$	• They're almost the same but the signs of the imaginary parts are different.		
» What if you plotted these numbers on an Argand Diagram? » Describe the position and the relationship between the two points	• They would be mirror images of each other. • One would be above the Real axis and the other below it. • It's an Axial Symmetry • $5 - 2i$ is a reflection of $5 + 2i$ in the Real axis and vice versa.	» Write the words **Reflection (Axial Symmetry) in the Real Axis** in the word bank.	» Can students visualise these numbers and recall the transformation? » If students are having difficulty, encourage them to draw it out.

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» We call these Complex Numbers conjugate pairs. » If $z = x + iy$, what is its complex conjugate? » We call $x - iy$ 'the conjugate' of $x + iy$ » Use the word conjugate to describe the previous examples.	• $x - iy$ • The conjugate of 5 + $2i$ is 5 – $2i$. • The conjugate of 5 - $2i$ is 5 + $2i$. • The conjugate of 3 – $7i$ is 3 + $7i$. • The conjugate of 3 + $7i$ is 3 – $7i$.		
» We use this symbol $\bar{z}$ to represent the conjugate of a Complex Number.		» Write on the board the conjugate of $x + iy$ is $x - iy$. $\bar{z}$ stands for the conjugate of z.	
» Write in your own words a definition of complex conjugate. » Illustrate with two examples. When you have finished share your definition with your partner.		» Circulate as students write a definition of complex conjugate.	» Are students expressing their understanding of complex conjugate in their definition? If a student's definition is incomplete, get them to compare their definition with others in the class.
» Do you remember where you met this idea previously in your study of Complex Numbers?	• When we were solving quadratic equations that had a negative square root. • The roots of a quadratic equation with complex roots are conjugate pairs.		

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» If you were to plot the product of conjugate pairs on an Argand Diagram, where would the result be? » Why?	• On the real axis • When you multiply two Complex Numbers the angles they make with the real axis add up to 0°. • With conjugate pairs the angles have to be equal but of opposite sign. • The result has to be on the x – axis and therefore Real.		» Assess student participation in the answering of this question. If students are having difficulty visualising the concept get them to draw it out. » Can students see that algebraically and geometrically that it is a Real number? » Students can check their algebraic result against the geometric interpretation of adding angles and multiplying the moduli.
» We are now going to practise this using **Section F, Student Activity 1.**		» Distribute **Section F, Student Activity 1, page 48-49.** » Walk around the classroom and observe what students are writing. Assist them as required.	

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
Section G: Division of Complex Numbers			
» Calculate $\frac{6+2i}{2}$ » Does this make sense? » Now calculate $(6 + 2i) \div 5$ » What type of number are you dividing by?	• $3 + i$ • Yes, all of $6 + 2i$ is divided by 2 • $(6 \div 5) + (2i \div 5) = \frac{6}{5} + \frac{2}{5} i$ • You are dividing by a real number	» Write on the board $\frac{6+2i}{2}$ » Ask a student to write up the solution on the board in the form $a + ib$	» Are students dividing both real and imaginary parts by 2?
» How would we calculate $(6 + 2i) \div (5 + i)$?	• I don't know • How do you divide by a Complex Number?	» Allow students to adopt an investigative approach here. Delay giving the procedure.	
» What are the only types of numbers you know how to divide by?	• Real numbers		
» How could we get a REAL denominator?	• Multiply the denominator by its complex conjugate.	» Write in word bank: **Denominator**	
» We know that the value of $\frac{\text{any number}}{\text{any number}} = 1$ » Give me examples	• $\frac{5}{5} = 1$ • $\frac{x}{x} = 1$	» Write in the word bank **Numerator** $\frac{\text{any number}}{\text{any number}} = 1$ » Write examples offered by the students on the board.	

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» Explain in words using mathematical terms such as denominator and numerator why multiplying the numerator and denominator by the same number does not change its value.	• Multiplying the numerator and denominator by the same number is a multiplication by 1 and doesn't change the value of the outcome.		» Are students able to apply prior knowledge of complex conjugates, if not direct the students to the previous activity?
» What must we do to balance out the multiplying of the denominator by its conjugate?	• Multiply the numerator by the same number, the conjugate.		
» How will you make $\frac{6+2i}{2+3i}$ into an equivalent fraction, which will allow you to convert this into division by a real number? » Complete this calculation.	• Multiply the numerator and denominator by 2-3i.	» Circulate and check to see that students are able to carry out this procedure.	» Are students applying their knowledge of multiplying fractions and that $\frac{2-3i}{2-3i}=1$ So we are only multiplying by 1.
» Share with your partner what the key to dividing by a Complex Number is, using mathematical terms like conjugate, denominator, numerator, equivalent fraction. **Note:** The students are arriving at an understanding of the division 'rule' based on previous knowledge.	• Multiply the denominator by its complex conjugate. In order to make an equivalent fraction you must also multiply the numerator by the complex conjugate.		» Are students arriving at an understanding of the division 'rule' based on previous knowledge?

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Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» In pairs come up with a wording for "Division of Complex Numbers, Rule# 4" **SectionB, Student Activity 3,** using maths vocabulary: numerator, denominator and conjugate. » Illustrate with two examples.	• When you divide Complex Numbers you multiply the numerator and denominator by the conjugate of the denominator. This makes the denominator into a real number.	» Walk around classroom and observe student progress.	» Can students complete the writing task using the correct terminology?
» In **Section E, Student Activity 2** we multiplied $3 + i$ by $1 + 2i$ to get $1 + 7i$ and represented them on an Argand Diagram. » Now calculate $\frac{(1+7i)}{(1+2i)}$	• $\frac{(1+7i)}{(1+2i)}\frac{(1-2i)}{(1-2i)} = \frac{15+5i}{5} = 3+i$		
» Use the Argand Diagram in **Section E, Student Activity 1** to investigate what is happening geometrically when we divide Complex Numbers.	• The modulus of the numerator is divided by the modulus of the denominator. • The angles for $1 + 7i$ and $1 + 2i$ subtract to give the angle for $3 + i$.	» Write on the board: **Subtract the Angles (Arguments)** **Divide the Moduli**	

Student Learning Tasks: Teacher Input	Student Activities: Possible and Expected Responses	Teacher's Support and Actions	Checking Understanding
» In pairs do **Section G, Student Activity 1**.		» Distribute **Section G, Student Activity 1, page 50.** » Circulate and check students' work, ensuring that all students can complete the task. Ask individual students to do questions on the board when the class has completed some of the work. They should explain why they are doing in each step.	
» Review your rule card and summarise Complex Number operations. » Read back over the sentence you wrote at the beginning of class on what you were going to learn today. How did you do? Now rate your performance from 1 to 5 adding a comment to explain your rating.	• Addition and subtraction are given in terms of real and imaginary parts. • Multiplication and division are in terms of angles and moduli.		

Appendix 1

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Introduction to Complex Numbers Using simple equations to see the need for different number systems

1.NATURAL NUMBERS (N)

N = 1,2,3,4,...are positive whole numbers.

Solve the following equations using only Natural Numbers:

(a) $x + 2 = 5$
(b) $x + 5 = 2$

Solution (a):

$x + 2 = 5$
$x + 2 - 2 = 5 - 2$
$x = 3$

The solution: 3 is an element of N or $3 \in N$.

Solution (b)

$x + 5 = 2$
$x = 2 - 5$
$x = ?$

We have a problem, since there is no natural number solution for 2 - 5. Therefore we must invent new numbers to solve for 2 – 5.

2.INTEGERS (Z)

Z = ...-3,-2,-1, 0, 1, 2, 3,...are positive and negative whole numbers.

Solve the following equations using only Integers:

(a) $x + 5 = 2$
(b) $3x = 4$

Solution (a):

$x + 5 = 2$
$x + 5 - 5 = 2 - 5$
$x = -3$

The solution: – 3 is an element of Z or $-3 \in Z$.

Solution (b)

$3x = 4$
$x = 4 \div 3$
$x = ?$

We have a problem since there is no integer which solves $4 \div 3$. Therefore we must invent new numbers to solve $4 \div 3$.

3.RATIONAL NUMBERS (Q)

These are numbers that can be written in the form a/b (fraction) where $a, b \in Z$ and $b \neq 0$.

Q =...-4.6, -4, -3.5, -2.07, -1, 0, 0.82,...
Q =... $-\frac{46}{10}$, $-\frac{4}{1}$, $-\frac{7}{2}$, $-\frac{207}{100}$, $-\frac{1}{1}$, $\frac{0}{1}$, $\frac{82}{100}$...

All repeating decimals can be written as rational numbers:

$0.\dot{3} = 0.333... = \frac{1}{3}$
$0.1\dot{6} = 0.1666... = \frac{1}{6}$
$0.\dot{1}4285\dot{7} = 0.142857142857... = \frac{1}{7}$

Solve the following equation using only Rational Numbers:

(a) $3x = 4$
(b) $x^2 = 5$

Solution (a)

$3x = 4$
$x = \frac{4}{3}$
Solution: $\frac{4}{3}$ is an element of Q, or $\frac{4}{3} \in Q$.

Solution (b)

$x^2 = 5$
$x = \sqrt{5}$
$x = ?$

We have a problem, since there is no rational number for the $\sqrt{5}$. Therefore we must invent new numbers to solve $\sqrt{5}$.

Appendix 1

Project Maths Tionscadal Mata Development Team

Introduction to Complex Numbers Using simple equations to see the need for different number systems (continued)

IRRATIONAL NUMBERS
These are numbers that cannot be written in the form a/b where $a, b, \in Z$ and $b \neq 0$. Irrational numbers are non terminating, non repeating decimals such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, π, e. Pythagoras came across the existence of these numbers around 500 BC.

$c^2 = a^2 + b^2$
$c^2 = 1^2 + 1^2$
$c^2 = 2$
$c = \sqrt{2}$

4.REAL NUMBERS (R)

This is the number system we get when we put all the Rational Numbers together with all the Irrational Numbers. The Rationals and Irrationals form a continuum (no gaps) of Real Numbers provided that the Real Numbers have a one to one correspondence with points on the Number Line.

Solve the following equations using Real Numbers:

$x^2 - 1 = 0$

$x^2 - 1 = 0$	or	$x^2 - 1 = 0$
$(x - 1)(x + 1) = 0$		$x = \pm\sqrt{1}$
$x = 1$ or $x = -1$		$x = \pm 1$

No problem since $-1, +1 \in R$

$x^2 - 3 = 0$
$x = \pm \sqrt{3}$
No problem since $-\sqrt{3}, +\sqrt{3} \in R$

Solve this equation using only Real Numbers

$x^2 + 1 = 0$
$x^2 = -1$
$x = \sqrt{-1}$

What number when multiplied by itself (squared) gives -1? What does your calculator say when you try it: ERROR. We have a problem since there is no Real Number for $\sqrt{-1}$. A number whose square is negative cannot be Real. Therefore we must invent new numbers to solve $\sqrt{-1}$.

5.COMPLEX NUMBERS (C)

Complex Numbers can be written in form
$z = a + ib$, where $a, b \in R$
$i^2 = -1$ and $i = \sqrt{-1}$
$\text{Re}(z) = a$ and $\text{Im}(z) = b$

Appendix 2

A History of Complex Numbers

Cardano and Tartaglia

Complex Numbers arose from the need to solve cubic equations and not as is commonly believed from the need to solve quadratic equations. Gerolamo Cardano and Tartaglia in the 16th century devised formulae to solve cubic equations of a "REDUCED" form. In their solution to cubic equations with Real roots which could be guessed, the square root of negative numbers turned up in the derivation of these solutions and there was no way to avoid them.

Solve the cubic equation:

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x - 1)(x + 1) = 0$

$x = 0, \quad x = 1, \quad x = -1.$

But when Tartaglia used his formula he got this solution:

$$x = \frac{1}{\sqrt{3}}\left(\left(\sqrt{-1}\right)^{\frac{1}{3}} + 1/\left(\sqrt{-1}\right)^{\frac{1}{3}}\right)$$

At first this looks like nonsense; however, formal calculations with Complex Numbers show that

$z^3 = i$

$z = \left(\sqrt{-1}\right)^{\frac{1}{3}}$ the cube roots of i has solutions

$$z = -i, \quad z = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Substituting these in for $z = \left(\sqrt{-1}\right)^{\frac{1}{3}}$ in Tartaglia's cubic formula and simplifying we get the solutions

$x = 0, \quad x = 1, \quad x = -1.$

This forced an investigation on the square root of negative numbers which has continued to the present day.

Imaginary Numbers

Descartes in the 17th century decided to call them "imaginary" numbers and, unfortunately, the name has stuck even though they are no more imaginary than negative numbers or any other numbers. Euler (around 1777) decided to give the name i to the number which is $\sqrt{-1}$ so now we have:

$i = \sqrt{-1}$ and $i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$

This was to avoid confusion like this:

$$\begin{aligned} 2 &= 1 + 1 \\ &= 1 + \sqrt{1} \\ &= 1 + \sqrt{-1, -1} \\ &= 1 + \sqrt{-1}.\sqrt{-1} \\ &= 1 + i.i \\ &= 1 + i^2 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

But $2 \neq 0$ So where is the mistake?

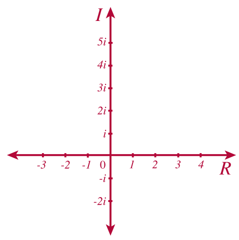
$\sqrt{ab} = \sqrt{a}.\sqrt{b}$ when $a \geq 0$ and $b \geq 0$ $a, b \in R$

or $a \geq 0$ and $b \leq 0$

or $a \leq 0$ and $b \geq 0$

But $\sqrt{ab} \neq. \sqrt{a}.\sqrt{b}$ when $a < 0$ and $b < 0$

In 1806, Argand published a way of representing Real Numbers and Imaginary Numbers on a diagram using two axes at right angles to each other like the Cartesian Plane; this is called the Argand Diagram.

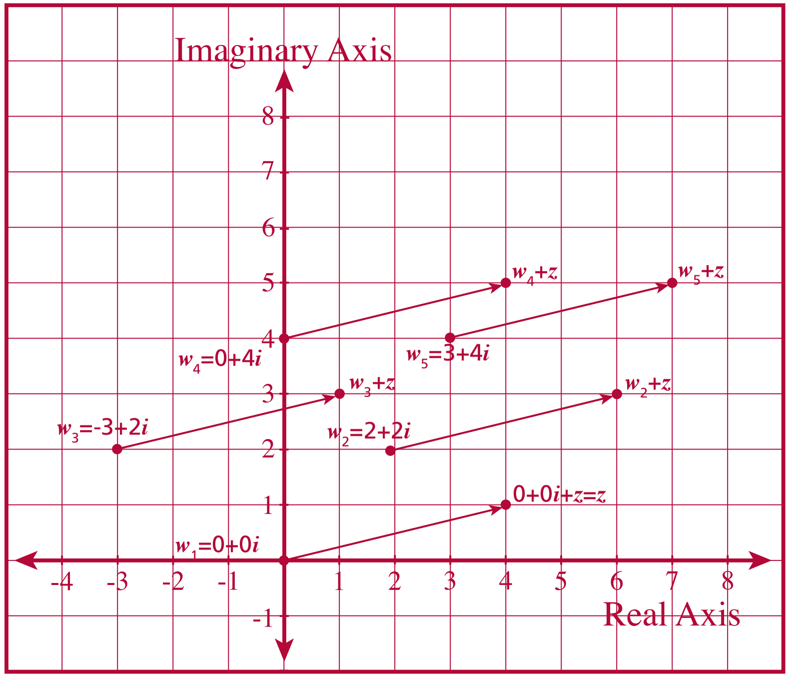


On the face of it, the Argand Diagram appears to be identical to the Cartesian Plane. However, it is quite different in that the points which are Complex Numbers can be added, subtracted, multiplied and divided in ways that cannot be done to the points in the Cartesian Plane.

Complex Numbers were further developed in 1843 by William Rowan Hamilton, who invented Quaternions: $i^2 = j^2 = k^2 = ijk = -1$ which extended the study of Complex Numbers to 3-D.

Appendix 3

Board Plan: The Addition and Subtraction of Complex Numbers

Word Bank	Algebra Method	Rules	Geometry Method
Real **Imaginary** **Translation** $\vec{oz}$ **on** w **Brackets** **Subtract ALL terms** **Translation** $\vec{zo}$ **on** w	$z = 4 + i$ and $w_1 = 2 + 2i$ $w_1 + z$ $= (2 + 2i)+(4 + i)$ $= 6 + 3i$ $w_1 - z$ $= (2 + 2i) - (4 + i)$ $= 2 + 2i - 4 - i$ $= -2 + i$	**Addition of Complex Numbers: Re + Re and Im + Im** $(a + bi) + (c + di)$ $= (a + c) + i(b + d)$ **Subtraction of Complex Numbers: Re – Re and Im – Im** $(a + bi) - (c + di)$ $= (a - c) + i\,(b - d)$	

Section A, Student Activity 1

Assignment

To consolidate their learning students could complete the student activities that follow and/or work together on completing the following tasks:

You have been hired to write an introduction to the section on Complex Numbers for The Project Maths Textbook for Leaving Certificate. Explain each number system and illustrate each explanation with an example that is not found in the reference documents.

Or

Prepare a visual display to represent the various Number Systems

Section A, Student Activity 2

Number Systems

Question 1: Write down two examples of the following:

1. Natural numbers	
2. Positive integers	
3. Negative integers	
4. Integers	
5. Rational numbers which are positive and not natural numbers	
6. Rational numbers which are also integers	
7. Irrational numbers	
8. Real numbers	
9. Real numbers which are natural numbers, integers, and rational numbers	
10. What type of numbers are represented in the gap between set Z and set N i.e. Z \ N?	

Section A, Student Activity 2

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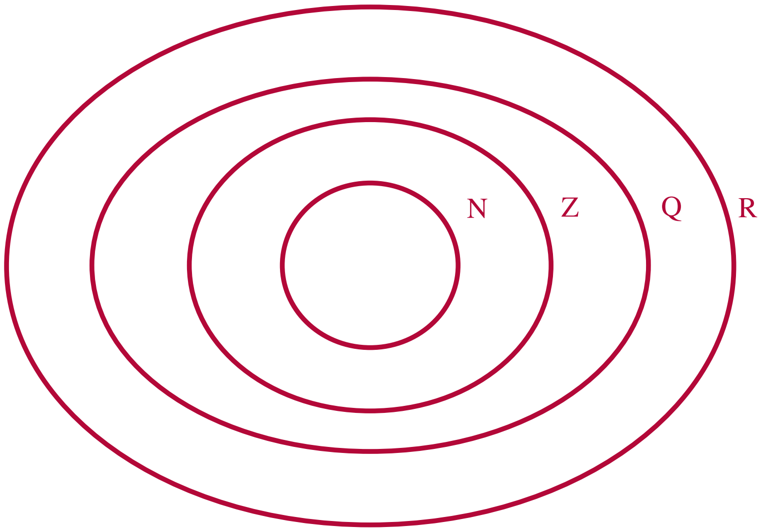
Number Systems (continued)

Question 2: Place the following numbers in their correct positions in the Venn Diagram below:

5, $\sqrt{5}$, $\frac{15}{3}$, 5.2, $-\frac{2}{3}$, 0.3333333......., 0.272727........., $\sqrt{4}$, π, -3



Question 3: Solve the following equations and then answer the questions that follow:

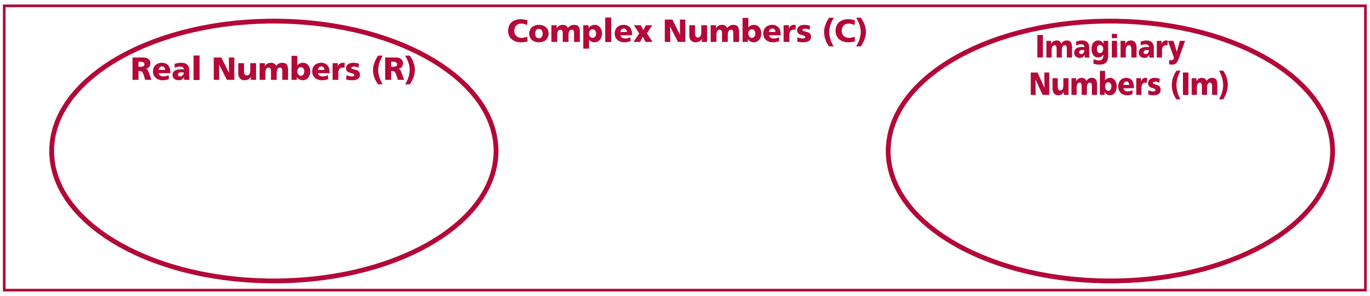
1. Solve $x + 2 = 5$ What type of numbers do you need to solve $x + 2 = 5$?	
2. Solve $x + 7 = 2$ What type of numbers do you need to solve $x + 7 = 2$?	
3. Solve $4x = -3$ What type of numbers do you need to solve $4x = -3$?	
4. $2x + \sqrt{3} = 0$ What type of numbers do you need to solve $2x + \sqrt{3} = 0$?	
5. What types of equations are the above? How many roots / solutions have they?	

Section A, Student Activity 2

Number Systems (continued)

Question 4: Can a number be real and imaginary at the same time? Can it be either? Place each of these numbers into the appropriate sets below: Imaginary number set, Real number set, Complex Number set

$$\left\{3,\ 0,\ 2+7i,\ 4+0i,\ -5+7i,\ \frac{2}{3}+5i,\ 0+2i,\ i,\ 7-\frac{4}{11}i,\ 5+6i,\ 9,\ 0-\frac{2}{3}i\right\}$$



On graph paper, plot each of the above Complex Numbers on an Argand Diagram. Complete the table below.

Complex Number	Real part	Imaginary part
3		
0		
$2+7i$		
$4+0i$		
$-5+7i$		
$\frac{2}{3}+5i$	$\frac{2}{3}$	5
$0+2i$		
i		
$7-\frac{4}{11}i$		
$5+6i$		
9		
$0-\frac{2}{3}i$		

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Section A, Student Activity 3

Powers of i

1. Simplify i^{11} Which answer is correct: ☐ 1 ☐ i ☐ -1 ☐ $-i$ Explain:	5. Simplify $4i^3 + 7i^9$ Which answer is correct: ☐ $11i$ ☐ $3i$ ☐ $-3i$ ☐ -11 Explain:
2. Simplify i^{33} Which answer is correct: ☐ 1 ☐ i ☐ -1 ☐ $-i$ Explain:	6. Simplify $(3i^5)^2$ Which answer is correct: ☐ -9 ☐ $-9i$ ☐ 6 ☐ 9 Explain:
3. Simplify $i^{16} + i^{10} + i^8 - i^{14}$ Which answer is correct: ☐ 0 ☐ 1 ☐ 2 ☐ i Explain:	7. Make up a similar question of your own and explain your answer.
4. Simplify $i^{12}.\ 3i^2.\ 2i^8$ Which answer is correct: ☐ $6i$ ☐ -6 ☐ $-6i$ ☐ 6 Explain:	8. Make up a similar question of your own and explain your answer.

Section A, Student Activity 4

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Solving Quadratic Equations

Quadratic Equation	$ax^2 + bx + c = 0$	Solving using the formula (see tables)	Roots
$x^2 + 6x + 13 = 0$	$a =$ $b =$ $c =$ $b^2 - 4ac =$		
$x^2 - 4x + 13 = 0$	$a =$ $b =$ $c =$ $b^2 - 4ac =$		
$2x^2 - 2x + 5 = 0$	$a =$ $b =$ $c=$ $b^2 - 4ac =$		

Section A, Student Activity 4

Solving Quadratic Equations (continued)

Quadratic Equation	$ax^2 + bx + c = 0$	Solving using the formula (see tables)	Roots
$x^2 - 10x + 34 = 0$	$a =$		
	$b =$		
	$c =$		
	$b^2 - 4ac =$		
$3x^2 - 4x + 10 = 0$	$a =$		
	$b =$		
	$c =$		
	$b^2 - 4ac =$		
$x - \frac{5}{x} = 3$	$a =$		
	$b =$		
	$c =$		
	$b^2 - 4ac =$		

Section A, Student Activity 5

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The Modulus of a Complex Number

You will need graph paper with this activity.
Use a different Argand Diagram with labelled axes for each question.

1. What is meant by the absolute value or modulus of $z = 5 + 2i$?

   ____

   Plot z on an Argand Diagram. Write z as an ordered pair of real numbers:

   ____

   ____

   Calculate $|z|$

2. Plot $-4i$ on an Argand Diagram. Write $-4i$ as an ordered pair of real numbers.

   ____

   ____

   Find the distance from (0, 0) to the number $-4i$?

3. Plot as accurately as you can the Complex Number $z = \sqrt{3} + 3i$
   Write this Complex Number as an ordered pair of real numbers.

   ____

   ____

   Calculate $|z|$

Section A, Student Activity 5

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The Modulus of a Complex Number (continued)

4. Find the modulus of the Complex Number $z = a + i\,b$.

   ____

   ____

   ____

   Summarise how you get the modulus or absolute value of a Complex Number by explaining what you do to the real and imaginary parts of the Complex Number.

   ____

   ____

   ____

   ____

5. Plot the point $3 + 4i$ on an Argand Diagram. Calculate $|3 + 4i|$
   Give the coordinates of 7 other points which are the same distance from the origin.

   ____

   Plot these points on an Argand Diagram.

   What geometric figure contains all the points which are this same distance from the origin?______________________________________

   Draw it on the Argand Diagram.

Section A, Student Activity 6

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What do I know and what do I need to learn?

Put a tick in the box that best describes each statement for you.

Knowledge	Yes	Uncertain	No
I know the number systems N,Z,Q,R and can perform the operations of +, −, ÷, x			
I can square numbers			
I can find the square root of numbers			
I know the rules of indices			
I know the rules governing surds (irrational numbers)			
I can add and subtract like terms			
I can multiply and simplify algebraic expressions with two terms			
I can measure with a ruler			
I can use a protractor			
I can use a compass			
I understand what happens when a positive whole number is multiplied by (i) a number > 1 and (ii) a number between 0 and 1			
I know how to solve linear equations			
I know how to solve quadratic equations			
I can apply Pythagoras's theorem			
I can find the measure of an angle in a right angled triangle when I know two lengths			
I know the two components of a Complex Number			
I know what i means			
I understand if i is raised to any power the result will be an element of the set $\{-1,1,i,-i\}$			
I know what letters are used to denote Complex Numbers			
I know how to visually represent Complex Numbers			
I know what a translation is			
I know the definition of an angle (Rotation)			
I know what an axial symmetry is			
I know what the modulus of a Complex Number is			
I can calculate the modulus of a Complex Number			

Section B, Student Activity 1

Rule Card

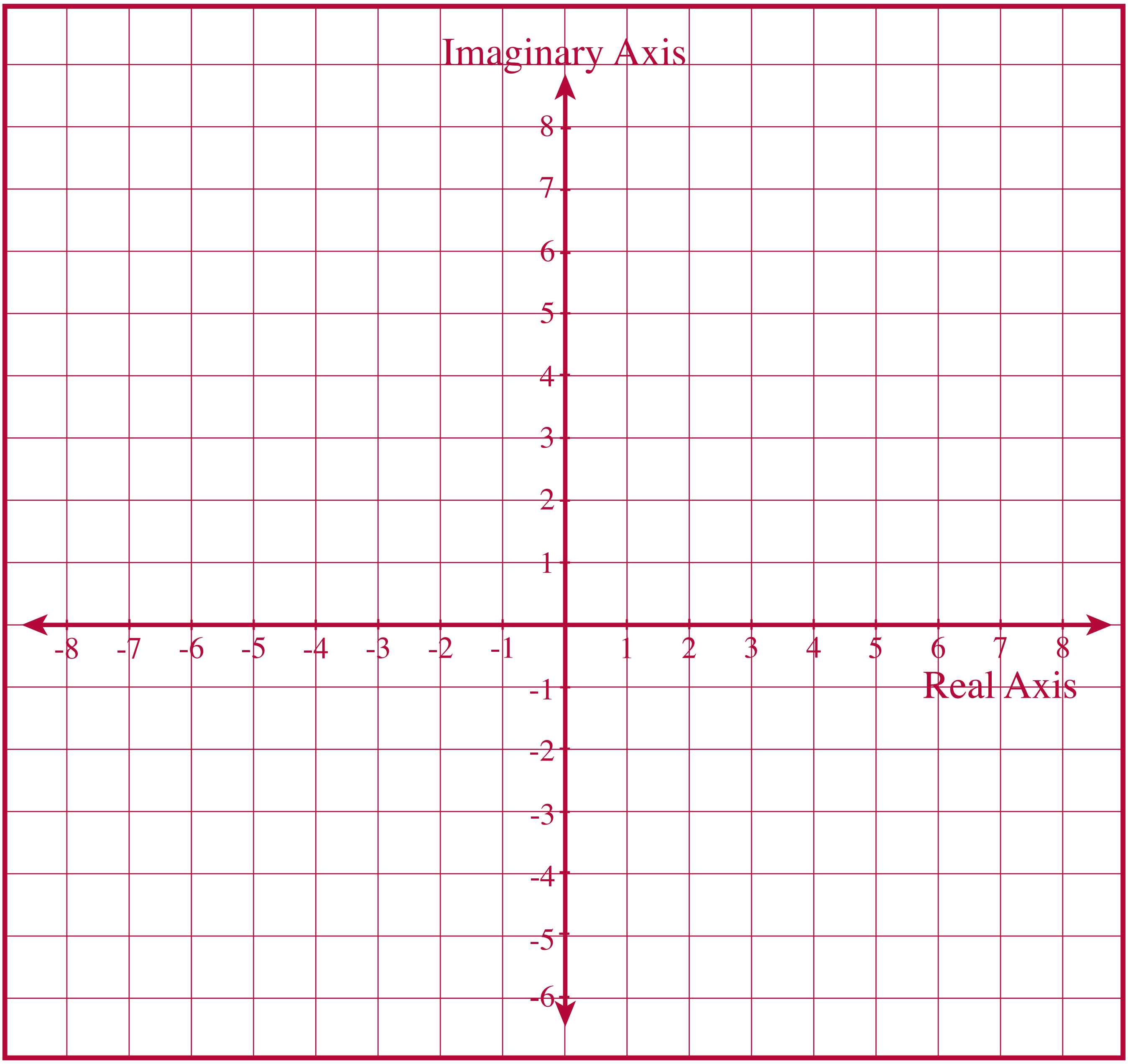
Addition	Subtraction	Multiplication	Division
Rule # 1	Rule # 2	Rule # 3	Rule # 4
Example 1:	Example 1:	Example 1:	Example 1:
Example 2:	Example 2:	Example 2:	Example 2:

Section B, Student Activity 2

Addition and Subtraction of Complex Numbers

1. Add $z = 4 + i$ to each of the following complex numbers:
   o $= 0 + 0i$
   $w_1 = 2 + 2i$
   $w_2 = -3 + 2i$
   $w_3 = 0 + 4i$

2. Represent the complex numberso, w_1, w_2, w_3, as points on an Argand Diagram and then show the results from the above exercise using a directed line (a line with an arrow indicating direction) between each w and its corresponding $w + z$. What do you notice?



Section C, Student Activity 1

Adding and Subtracting Complex Numbers: Practice Questions

1	$(12 + 4i) + (7 - 11i)$	
2	$(7 - 2i) + (9 - 4i)$	
3	$(4 - 6i) + (-5 - i)$	
4	$(3 - 8i) - (2 - 4i)$	
5	$(-12 - 5i) - (-2 - 8i)$	
6	$\left(2+\frac{1}{3}i\right)+\left(3-\frac{5}{6}i\right)$	
7	$\left(4+\sqrt{-16}\right)+\left(-5-\sqrt{-25}\right)$	
8	$z_1 = 5+i$ $z_2 = -4 + 6i$ $z_3 = -11 + 2i$ Calculate $(z_1 + z_2) - z_3$	
9	$\left(4-\sqrt{-50}\right)-\left(3+\sqrt{-8}\right)$	
10	$z_1 = a + bi$, $z_2 = c + di$ $z_1 + z_2 =$ $z_2 + z_1 =$ $z_1 - z_2 =$ $z_2 - z_1 =$	

Section D, Student Activity 1

Multiplication by a Real Number

1. If $z = 3 + 4i$, what is the value of $2z$, $3z$, $5z$, $10z$?
2. Represent the origin o = $0 + 0i$, z and $2z$ on an Argand diagram.
3. Find the distance z and $2z$ from o, the origin. Describe at least two methods.
4. Comment on your results.
5. Calculate and plot on an Argand Diagram $\frac{z}{2}, \frac{3z}{4}$. What do you notice?

Section E, Student Activity 1

Multiplication by an Imaginary Number

1. If $z = 3 + 4i$, what is the value of iz, i^2z, i^3z, i^4z? Represent your results on an Argand Diagram joining each point to the origin o = 0 + 0i.
2. Investigate what is happening geometrically when z is multiplied by i to get iz? Use geometrical instruments and/or calculation to help you in your investigation.
3. Prove true for the multiplication of iz by i that you get i^2z and the multiplication of i^2z by i that you get i^3z etc.
4. Write your conclusion.
5. Plot on an Argand Diagram $4 + 2i$ and $-i$. Multiply $-i\,(4 + 2i)$. What do you notice?

Section E, Student Activity 2

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Multiplication of Complex Numbers in the form $a + ib$

1. Plot $3 + i$, $1 + 2i$ and their product $1 + 7i$ on an Argand Diagram.
2. Join each point to the origin $o = 0 + 0i$.
3. Measure the angle (by instrument or calculation) made by the line joining $3 + i$ to the origin and the Real Axis and likewise for $1 + 2i$ and $1 + 7i$.
4. What do you notice about the angles?
5. Find the modulus of $3 + i$, $1 + 2i$ and $1 + 7i$.
6. What do you notice?

Section E, Student Activity 3

Multiplying Complex Numbers

When multiplying Complex Numbers all answers are to be given in the form $a+ib$

1	a. Multiply $-4 + 3i$ by 2. b. Plot $-4 + 3i$ and $2(-4 + 3i)$ on an Argand Diagram. c. Calculate $\lvert -4 + 3i\rvert$ and $\lvert 2(-4 + 3i)\rvert$. d. What was the effect of multiplication by 2 on $-4 + 3i$?	
2	a. Multiply $-4 + 3i$ by i. b. Plot $-4 + 3i$ and $i(-4 + 3i)$ on an Argand Diagram. c. Calculate $\lvert i(-4 + 3i)\rvert$. d. What was the effect of multiplication by i on $-4 + 5i$?	
3	a. Plot $4 + 2i$ and $-i$ on an Argand Diagram. b. Multiply $-i(4 + 2i)$. c. Plot $-i(4 + 2i)$ on an Argand Diagram. d. What was the effect of multiplication by $-i$ on $4 + 2i$?	
4	a. Plot $1 + i$ on Argand Diagram. b. Calculate $\lvert 1 + i\rvert$ c. What angle does the line segment joining $1 + i$ to the origin make with the positive direction of the x axis?	

Section E, Student Activity 3, (continued)

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Multiplying Complex Numbers

4	d. Using what you know about multiplication of one Complex Number by another, what 2 transformations will happen to $1 + i$ if it is multiplied by $(1 + i)$? e. Knowing the modulus of $1 + i$ and the angle it makes with the Real axis, use this information to work out $(1 + i)(1 + i)$. f. Now calculate $(1 + i)(1 + i)$ multiplying them out as you normally would. g. Were you correct in your first answer?	
5	a. Plot $1 + 6i$ and $-1 - 2i$ on an Argand Diagram. b. Multiply $(1 + 6i)(-1 - 2i)$. c. Plot the answer on an Argand Diagram.	
6	If $z_1 = (5 + 4i)$ $z_2 = (3 - i)$ a. Plot z_1 and z_2 on an Argand Diagram. b. Calculate $z_1 . z_2$. c. Plot the answer $z_1 z_2$ on an Argand Diagram.	
7	a. Plot $3 - 2i$ and $3 + 2i$ on an Argand Diagram. What do you notice about both points? b. What angle do you expect the product $(3 - 2i)(3 + 2i)$ to make with the x - axis? Explain. c. Multiply $(3 - 2i)(3 + 2i)$. What do you notice about the answer?	
8	a. Plot $4 + 3i$ on an Argand Diagram b. Plot $(4 + 3i)^2$ on an Argand Diagram	
9	a. Plot on an Argand Diagram: $5 + i(4 - 2i)$	
10	If $z_1 = a + bi$ and $z_2 = c + di$, then a. $z_1 . z_2 =$ b. $z_2 . z_1 =$	

Section F, Student Activity 1

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Complex Conjugate

For all questions z_1= -5 + 4i and w_1 = 3 - 3i

1	What is $\bar{z}_1$?	
2	What is $z_1 + \bar{z}_1$?	
3	What is $\bar{w}_1$?	
4	What is $w_1 + \bar{w}_1$?	
5	If $z = a + i\,b$, what is $\bar{z}$?	
6	Calculate $z + \bar{z}$ What type of number is $z + \bar{z}$?	
7	What can you say about 2 Complex Numbers if the sum of the 2 Complex Numbers is real?	
8	What conclusion can you make about the sum of a Complex Number and its conjugate?	
9	Calculate $z_1 - \bar{z}_1$	
10	Calculate $w_1 - \bar{w}_1$	
11	If $z = a + ib$, what is $\bar{z}$?	
12	Calculate $z - \bar{z}$ What type of number is z- $\bar{z}$?	
13	What conclusion can you make about the difference between a Complex Number and its conjugate? i.e. $z- \bar{z}$	
14	Calculate $z_1 \,.\, \bar{z}_1$	
15	Calculate $w_1 \,.\, \bar{w}_1$	
16	If $z = a + ib$, what is $\bar{z}$?	
17	Calculate $z\,\bar{z}$	
18	What type of number is $z\,\bar{z}$?	
19	When you multiply a Complex Number by its conjugate what type of number do you get?	

Section F, Student Activity 1

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Complex Conjugate (continued)

20	Remember that when you multiply two Complex Numbers you rotate one of them by the angle the other one makes with the positive direction of the x axis and you stretch the length (modulus) of one by the modulus of the other. Check that this makes sense for $z_1\ \bar{z}_1$ a. Plot z_1 and $\bar{z}_1$ on an Argand Diagram. Join each point back to the origin (o). b. Measure the angle made by oz_1 and the positive direction of the x –axis (θ_1) c. Measure the angle made by o$\bar{z}_1$ and the positive direction of the x –axis. (θ_2) Remember angles measured in an anticlockwise direction from the x –axis are positive and angles measured in clockwise direction from the x –axis are negative d. Rotate oz_1 by θ_2 e. What is $\theta_1 + \theta_2$? f. Multiply $\|z_1\|\ \|\bar{z}_1\|$ g. Compare the combined transformations of rotating and stretching with $z_1\ \bar{z}_1$.	

Section G, Student Activity 1

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Division of Complex Numbers

Write the following in the form $a + ib$

1.	$\frac{9-6i}{3}$	
2.	$\frac{1}{i}$	
3.	$\frac{7-4i}{1-2i}$	
4.	$\frac{3+i}{3-i}$	
5.	$\frac{2-4i}{-i}$	
6.	$\frac{1}{5-4i}$	
7.	$(5-4i)\left(\frac{5}{41}+\frac{4i}{41}\right)$	
8.	**Find the multiplicative inverse of 3 - 2i**	
9.	**Calculate the quotient of** $\frac{1+7i}{1+2i}$ **Plot 1 + 7i, 1 + 2i and their quotient on an Argand Diagram** **Calculate | 1 + 7i |, | 1 + 2i |,|quotient| and investigate if** $\frac{\lvert 1+7i\rvert}{\lvert 1+2i\rvert}$ = **|quotient|.** **Calculate the angles that 1 + 7i and 1 + 2i make with the Real axis and investigate if the subtraction of these angles is equal to the angle that the quotient makes with the Real axis.**	
10.	$z_1 = a + bi$, $z_2 = c + di$, **Find:** $\frac{z_1}{z_2} =$ $\frac{z_2}{z_1} =$	