Teaching & Learning Plans

Introducing *e*

Leaving Certificate Syllabus
Higher level

Project Maths
Tionscadal Mata
Development Team

The Teaching & Learning Plans are structured as follows:

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Aims outline what the lesson, or series of lessons, hopes to achieve.

Prior Knowledge points to relevant knowledge students may already have and also to knowledge which may be necessary in order to support them in accessing this new topic.

Learning Outcomes outline what a student will be able to do, know and understand having completed the topic.

Relationship to Syllabus refers to the relevant section of either the Junior and/or Leaving Certificate Syllabus.

Resources Required lists the resources which will be needed in the teaching and learning of a particular topic.

Introducing the topic (in some plans only) outlines an approach to introducing the topic.

Lesson Interaction is set out under four sub-headings:

i. **Student Learning Tasks – Teacher Input:** This section focuses on possible lines of inquiry and gives details of the key student tasks and teacher questions which move the lesson forward.

ii. **Student Activities – Possible Responses:** Gives details of possible student reactions and responses and possible misconceptions students may have.

iii. **Teacher's Support and Actions:** Gives details of teacher actions designed to support and scaffold student learning.

iv. **Assessing the Learning:** Suggests questions a teacher might ask to evaluate whether the goals/learning outcomes are being/have been achieved. This evaluation will inform and direct the teaching and learning activities of the next class(es).

Student Activities linked to the lesson(s) are provided at the end of each plan.

Teaching & Learning Plans: Introducing e

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Aims

- To introduce the number e as the base rate of growth for all continually growing processes

Prior Knowledge

Prior knowledge and experience of handling fractions and percentages is required.

Students have prior knowledge of

- Patterns with numbers
- Exponential functions such as $y = a2^x$, $y = a3^x$, where $a \in N, x \in R$
- Indices
- Compound interest and the compound interest formula
- Logarithms

Learning Outcomes

As a result of studying this topic, students will be able to

- Link continuously compounded interest and the number e
- Understand the relationship between e and the natural logarithm ($\log_e$)

Real Life Context

The following examples could be used to explore real life contexts.

- Continuously compounding interest
- Bacterial growth
- Radioactive decay
- Rate of chemical reaction

Relationship to Leaving Certificate Syllabus

Sub-Topic	
Students learn about	**In addition students working at HL should be able to**
5.1 Functions	– recognise surjective, injective and bijective functions – find the inverse of a bijective function – given a graph of a function sketch the graph of its inverse – express quadratic functions in complete square form – use the complete square form of a quadratic function to • find the roots and turning points • sketch the function – graph functions of the form • $ax^2 + bx + c$ where $a, b, c \in \mathbf{Q}$, $x \in \mathbf{R}$ • ab^x where $a, b \in \mathbf{R}$ • logarithmic • exponential • trigonometric – interpret equations of the form $f(x)$ and $g(x)$ as a comparison of the above functions – informally explore limits and continuity of functions

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Teacher Reflections

Lesson Interaction

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
Review of exponential functions			
» What exponential functions have you met to date?	• $y = ab^x$ where a represents the starting value, b is the constant rate of growth for a given time interval and x is the number of those time intervals. • In the pocket money example, if you started with €3, doubled your money each day you would have $3(2)^4$ euro after 4 days.	» Remind students of the pocket money example and the story surrounding the payment by the Grand Vizier in grains of wheat for the invention of the game of chess.	» Are students able to recognise that in exponential functions the variable is in the exponent and not in the base? » Do students know that when a function is exponential there is a constant called a growth factor and that during each time interval, the amount present is multiplied by this growth factor? » Can they recall that exponential functions such as $y = a2^x$, where $a \in N$, $x \in R$ are always increasing?
» What are the variables and constants in this equation?	• The starting value a and the rate of growth *b*, are constant and the variable x appears in the exponent.	» Write the equation $y = ab^x$ on the board.	» Are students able to distinguish the variable from the constants in an exponential function?
» Plot a graph of $y = 2^x$ $-2 \le x \le 5, x \in R$ » What is the effect of negative exponents? What if we worked out 2^{-1000}? » Is it possible for this function to give negative y values? **Note:** Point out the similarity between this and geometric sequences.	• Students draw up a table of values and plot the graph • Negative exponents lead to values of y between 0 and 1 but never negative values of y. • No. No value of x will give negative y.	» With *GeoGebra* draw $y = a^x$ and move the slider to change the value of a to highlight this. (See page 16 on how to do this using *GeoGebra*).	

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
» What do you notice about the scales you are using for the x and y axes for the graph of $y = 2^x$? Are they the same?	• You need to compress the scale on the y axis relative to the x axis scale. The y values start very small and are very close to 0 initially but once the y values start to grow they grow faster and faster. When x increases by 1 more than its previous value, y increases to twice more than it had been. There is a constant ratio between the output values for successive input values.	» Remind students of the constant difference between successive outputs for linear functions of the form $y = a + bx$ and the constant second differences for quadratic functions.	» Do students see that exponential functions grow quickly because of the multiplicative nature of the relationship? E.g. $2^{x+1} = 2.2^x$ $2^{x+2} = 2^2.2^x$
» What is meant by exponential growth when people speak about it in everyday terms?	• Growing slowly initially then growing faster and faster.		
» Can you think of any examples of exponential growth?	• Bacterial growth 2^x, population growth	» Tell students that population growth can approximate to, or be modelled by, exponential growth, but the growth factor in reality is not exactly constant.	» Are students able to come up with examples of exponential growth?
» Exponentials functions always have a positive number other than 1 as a base. What does this mean?	• b in the formula $y = ab^x$ is always positive		

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Teacher Reflections

<table>
<tr><th>Student Learning Tasks: Teacher Input</th><th>Student Activities: Possible Responses</th><th>Teacher's Support and Actions</th><th>Assessing the Learning</th></tr>
<tr><td>» Recall the compound interest formula. What do the variables signify?</td><td>• $F = P\ (1 + i)^t$
F = final value,
P = principal (starting value), i = annual interest rate,
t = time in years.</td><td>» Remind students to use their Formulae and Tables book.</td><td>» Are students familiar with the compound interest formula or do they need short revision on it?</td></tr>
<tr><td>» Compare this with the formula $y = ab^x$. What are the similarities?</td><td>• It is the same type of formula where $a = P$, the growth factor b is represented by $(1 + i)$ and the variable x is the number of periods of compounding.</td><td>» Inform students that the banks use different kinds of compounding schemes – yearly, half yearly, quarterly, monthly, weekly and daily.</td><td></td></tr>
<tr><td>» Let us have another look at our “doubling” function and relate it to the compound interest formula. Start with €1 at time $t = 0$, make a table to show the effect of doubling after each unit of time.</td><td><table><tr><th>Time t</th><th>Final value F</th></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>8</td></tr><tr><td>4</td><td>16</td></tr></table></td><td></td><td></td></tr>
<tr><td>» What is i in the above “doubling” function? Use the C.I. formula to work out each value?</td><td>• i = 100%
$t = 1$: $F = P\ (1 + i)^t = 1\ (1 + 100/100)^t = (1 + 1) = 2$
$t = 2$: $F = P\ (1 + i)^t = 1\ (1 + 1)^2 = 4$
$t = 3$: $F = P\ (1 + i)^t = 1\ (1 + 1)^3 = 8$
$t = 4$: $F = P\ (1 + i)^t = 1\ (1 + 1)^4 = 16$</td><td>» Write on the board $y = (1 + 1)^t$</td><td>» Do students see that the formula $y = 2^t$ and the compound interest formula with i = 100% and P = €1 give the same results?</td></tr>
</table>

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
» The compound interest formula assumes growth occurs in discrete steps, not continuously. Let $y = 2^x$ represent the growth of 1 bacterium after x periods of time. A new bacterium does not suddenly appear after 1 unit of time. It is continuously growing from $t = 0$ to $t = 1$. Let us see what would happen if we reduced the intervals of time for which interest was compounded so that it simulated natural growth which occurs continuously and not in jumps.		» Ask students where they've met "continuous" and "discrete" before?	» Are students associating the idea of continuous and discrete growth with continuous and discrete data which they encountered in statistics?
» Let us first split up the year into 2 periods so that interest is added every six months. We still start with €1 and instead of 100% interest per year we get 50% every 6 months. Use the C.I. formula to compute the final value at the end of the year.	• $F = P\ (1 + i)^t$ • $F = 1\ (1 + 1/2)^2 = 2.25$		
» We now have €2.25 after 1 year instead of €2 after 1 year with 100% interest added at the end of the year. Explain.	• After 6 months the €1 earned 50% interest giving €1.50 which then earned 50% interest giving €1.50 + €0.75 = €2.25.		
» If we now have interest added after 4 equal intervals in the year with interest at (100/4)% after each interval what is the final value at the end of 1 year?	• $F = 1\ (1 + 1/4)^4 =$ €2.44		» Can students see a pattern emerging i.e. the formula $F = 1\ (1 + 1/n)^n$?

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
» What do you notice about the final value as you compound more often?	• The final value is greater the shorter the time period used for adding on interest.		
» Will this trend continue? Investigate in pairs using **Student Activity 1**. » Distribute **Student Activity 1.** » Encourage students to look for patterns.	• Students fill in the table for **Student Activity 1**.	» Check the accuracy of students' work. » Allow students, who reach the limit of their calculators' computing ability, to use a spreadsheet.	» Did all students see their answers getting closer and closer to some "limit"? Did some students try to reduce the time further and reach the limit of their calculators' computing ability?
» What conclusion have you reached?	• As the time interval for compounding decreases this becomes more like continuous growth. • The more often we compound the greater the final value. However further increases in the number of compoundings per year seem only to cause changes in less and less significant digits. • The rate of growth if we continually compound 100% on smaller and smaller time intervals seems to be about 2.7182 (to four decimal places) as those digits do not change as we reduce the time.		

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
» This rate of growth is a number called e = 2.71828........ and is the base rate of growth shared by all continuously growing processes. It is a fundamental constant like π and it is irrational like π. What does this mean? » What does π represent?	• A number is irrational when it cannot be written in the form a/b where $a, b \in Z$. • Irrational numbers when expressed as decimals are non-terminating and non- repeating. • π is the ratio of the circumference of any circle to its diameter.		» Can students articulate what an irrational number is and what is meant by π? » Can they recall any other irrational numbers?
» We have only approximated e here as we haven't actually shown continuous growth – only for intervals of 1 second.			
» The number e shows up in population growth and in radioactive decay – in systems which exhibit continuous growth or decay.			
» Can you generalise the formula you were using where n represents the number of time intervals?	• $F = 1\ (1 + 1/n)^n$		
» If we find the limiting value of $F = 1\ (1 + 1/n)^n$ as n goes to infinity we have the number e. We write this as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ » The Swiss mathematician Euler was the first to use the notation e for this irrational number in 1731. » Use your calculators to find an approximate value for e.	• Using the calculator: $e^1 \approx 2.71828183$		

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
» Have you any comments on the interest rate we used?	• 100% interest – this is not realistic – no bank will ever give 100% interest.		
» What if growth was 50% per year (i= 0.5) instead of 100% on a principal of €1 ,given n compounding periods per year and the interest rate for each compounding period = i/n, how would the final value relate to e?	• A rate of 50% would give: $F = 1\ (1 + 0.5/n)^n$		
» Express F in terms of x by letting $x = \frac{n}{0.5}$ » First write n in terms of x.	$x = \frac{n}{0.5} \Rightarrow n = 0.5x$ and $\frac{0.5}{n} = \frac{1}{x}$ $\Rightarrow F = \left(1 + \frac{0.5}{n}\right)^n = \left(1 + \frac{1}{x}\right)^{0.5x}$		
» Assume that compounding is continuous – how will that affect n, the number of compounding periods? » How will n tending to infinity affect x? » What is the limiting value for F as x tends to infinity?	• n will tend to infinity. • As n = 0.5x, as n tends to infinity, x will also tend to infinity. $F = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{0.5x} = \lim_{x\to\infty}\left(\left(1 + \frac{1}{x}\right)^{x}\right)^{0.5}$ $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{x} = e \Rightarrow \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{0.5x} = e^{0.5}$ $F = e^{0.5}$		» Were students able to conclude that x would tend to infinity if n tends to infinity? » Did students spot e in the formula for the limiting value for F?

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
» For any yearly interest rate i, and n compounding periods per year, where i/n is the compound interest rate per period, can you write the final value at the end of the year in terms of e, assuming continuous compounding i.e. as n tends to infinity?	$x = \frac{n}{i} \Rightarrow n = ix$ $and\ \frac{i}{n} = \frac{1}{x}$ $\Rightarrow \left(1+\frac{i}{n}\right)^n = \left(1+\frac{1}{x}\right)^{ix}$ $F = \lim_{n\to\infty}\left(1+\frac{i}{n}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{ix}$ $F = e^i$		
» When the rate is 100% or 1, the final value after 1 year is e^1. When the rate is 50%, the final value after 1 year is $e^{0.5}$. Can you generalise what is happening?	• Final value after 1 year for continuous compounding of €1 when the yearly interest rate is i is e^i.		
» If the time is $t = 3$ years for instance, and continuous compounding occurs yearly at interest rate of i what is the final value?	• $F = (e^i)\ (e^i)\ (e^i) = e^{3i} = (e^i)^3$ $F = e^{\text{rate x time}}$ where r is the rate and t is the time.		
» The exponent of e in the above equation could be called $x = it$. » If x is 0.20 what could the possible whole number values of i and t be?	• i could be 20% for 1 year or 10% for 2 years or 5% for 4 years or 4 % for 5 years or 2% for 10. All will yield a final value of $e^{0.2}$ = 1.22 (to 2 d.p.) if €1 is invested and interest compounded continuously.		

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
» When we invest €1 at 5% continuous compound interest for t years the final value is $1e^{it}$ where $i = 0.05$. What is the final value if we invest €10 at 5% for t years?	• $F = e^{it}$ • $F = 10e^{0.05t}$		
» We saw that a 100% increase became approximately a 171.8% increase after 1 year of continuous compounding.			» Can students distinguish between the final value of 2.1718 and the increase of 1.1718?
» Let's see the difference continuous compounding and annual compounding makes for "normal" amounts of money, interest rates and times.			
» Calculate the final value if €5,000 is invested for 5 years at 3% per annum or if it is invested at 3% continuous compounding.	• $F = P\ (1 + i)^t$ • $F = 5{,}000\ (1 + 0.03)^3 = 5{,}796.37$ • $F = Pe^{it} = 5{,}000e^{(0.03)\ (5)} = 5{,}809.17$ • Difference = €12.80		
» Wrap up: What have you discovered about e?	• e is a fundamental constant like π. • It shows up as **the base growth rate for continuously growing systems.**		

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Teacher Reflections

Student Learning Tasks: Teacher Input	Student Activities: Possible Responses	Teacher's Support and Actions	Assessing the Learning
	• If we calculate the final value when €1 is invested at 100% compound interest for 1 year where the interest is compounded for increasingly smaller intervals we arrive at an approximation of e. • $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$		
Homework » Use a computer software package such as *GeoGebra* or *Excel* to plot graphs of $y = a\,e^x$. » How does the value of a affect the graph? » Plot a graph of $y = a\,e^{-x}$, $a > 0$. » What do you notice about the y values now compared to the y values for $y = a\,e^x$, $a > 0$	• a affects the rate of change of the function. When $a > 1$, the graph passes through (a, 0) and the larger a is the greater the rate of change of the function. • When a is negative the graph of $y = a\,e^x$ is a reflection of the graph of $y = a\,e^x$ in the x-axis when a is positive. • The y values are decreasing, very quickly initially and then slowing down.	» Use *GeoGebra* here to see that the slope of the tangent is increasing and reinforce the concept: $y = ae^x$ $dy/dx = ae^x$ The effect of a is that the slope increases by a multiple of a.	

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Teacher Reflections

Student Activity 1

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Invest €1 for 1 year at 100% compound interest.

Investigate the change in the final value, if the annual interest rate of 100% is compounded over smaller and smaller time intervals. (The interest rate i per compounding period will be calculated by dividing the annual rate of 100% by the number of compounding periods per year.)

Compounding period	Final value $F = P(1 + i)^t$ where i is the interest rate for a given period and t is the number of compounding periods per year. Calculate F correct to eight decimal places.
Yearly: $i = 1$	$F = 1(1+1)^1 = 2$
Every 6 months: $i = ½$	$F = 1\left(1+\frac{1}{2}\right)^2 =$
Every 3 months: $i =$ _______	
Every month: $i =$ _______	
Every week: $i =$ _______	
Every day: $i =$ _______	
Every hour: $i =$ _______	
Every minute: $i =$ _______	
Every second: $i =$ _______	
Conclusion:	

Student Activity 1 (Continued)

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Solutions

How often interst is compounded	Final Value
Yearly	$F=1(1+1)^1=2$
Every 6 months	$F=1\left(1+\frac{1}{2}\right)^2=2.25$
Every 3 months	$F=1\left(1+\frac{1}{4}\right)^4=2.44140625$
Every month	$F=1\left(1+\frac{1}{12}\right)^{12}=2.61303529$
Every week:	$F=1\left(1+\frac{1}{52}\right)^{52}=2.69259695$
Every day	$F=1\left(1+\frac{1}{365}\right)^{365}=2.71456748$
Every hour	$F=1\left(1+\frac{1}{8760}\right)^{8760}=2.71812669$
Every minute	$F=1\left(1+\frac{1}{525600}\right)^{525600}=2.71827923$
Every second	$F=1\left(1+\frac{1}{31536000}\right)^{315360000}=2.71828162$
Conclusion:	Conclusion: The final value gets bigger and bigger but the rate at which it is growing slows down and seems to be getting closer and closer to some fixed value close to 2.718.

GeoGebra Tutorial

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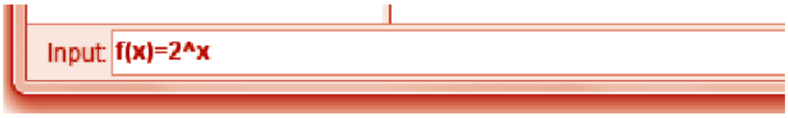
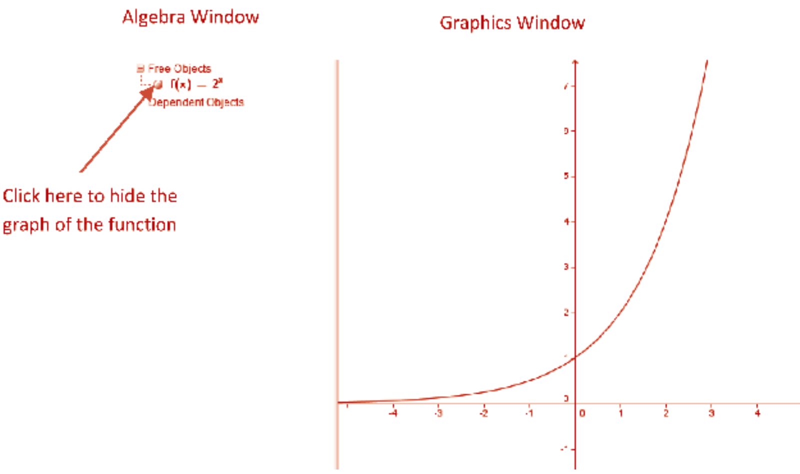
GeoGebra File to illustrate the sequence with general term $u_n = 2^n$

1. Plot the function $f(x) = 2^x$ in the **Input Bar**.

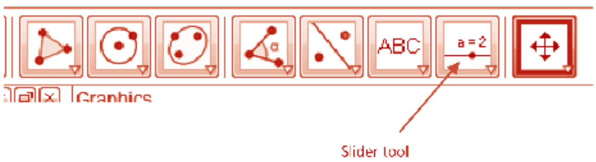


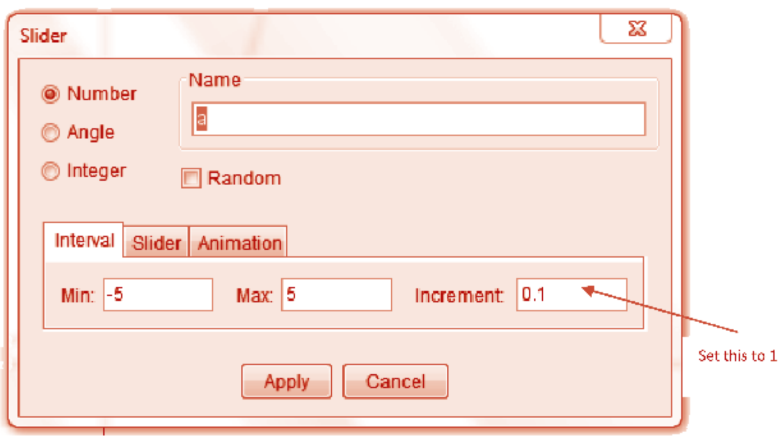
2. When you press **Return** the graph of the function is drawn. The diagram below shows the function drawn in **Graphics View** and the expression for the function in **Algebra View**.



3. Hide the graph of the function by clicking on the function button in **Algebra View**.

4. Create a slider by clicking on the **Slider Button** and then on **Graphics View**. In the dialogue box that then appears set the increments to 1 and click **Apply**.





GeoGebra Tutorial (Continued)

5. In the **Input Window** type (a, $f(a)$) and press **Return** and the corresponding point appears in **Graphics View.**

6. Right click on the point (a, $f(a)$) and select **"Show Trace"** from the dialogue box which results. Move the slider to show the locus of the point (a, $f(a)$).