Teaching through Problem Solving to promote Mathematical Thinking

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What are the purpose and the value of studying mathematics in primary and secondary schools?

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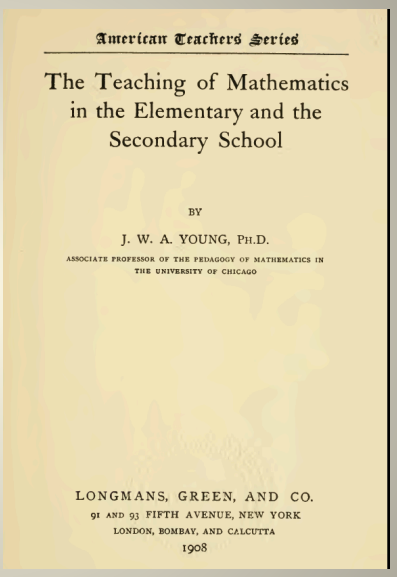
The facts of mathematics, important and valuable as they are, are not the strongest justification for the study of the subject by all pupils. Still more important than the subject matter of mathematics is the fact that it exemplifies most typically, clearly and simply certain modes of thought which are of the utmost importance to everyone.



American Teachers Series

The Teaching of Mathematics in the Elementary and the Secondary School

BY

J. W. A. YOUNG, PH.D.

ASSOCIATE PROFESSOR OF THE PEDAGOGY OF MATHEMATICS IN THE UNIVERSITY OF CHICAGO

LONGMANS, GREEN, AND CO.
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1908

Some pupils are tempted to evade precisely that portion of the work which gives the benefit, by memorizing the results of the work of others. This temptation is great to some pupils, and perhaps no other subject can become so barren and dreary as mathematics so studied. Ten pages of mathematics *understood* are better than a hundred memorized and not understood, and one page actually worked out independently is better than ten pages clearly but passively understood. The question is not *how much?* but *how?* The object is mastery, attainment of the spirit of the subject, and not to train the memory, or to ingest a large bulk of mathematical fact and formulas.

Memorizing.

(J.W.A. Young, 1908, p.38)

Thinking Mathematically

Mason, J., Burton, L., & Stacey, K, 1982

- You can think mathematically.
- Mathematical thinking can be improved by practice with reflection.
- Mathematical thinking is provoked by contradiction, tension and surprise.
- Mathematical thinking is supported by an atmosphere of questioning, challenging and reflecting.

Problem Solving

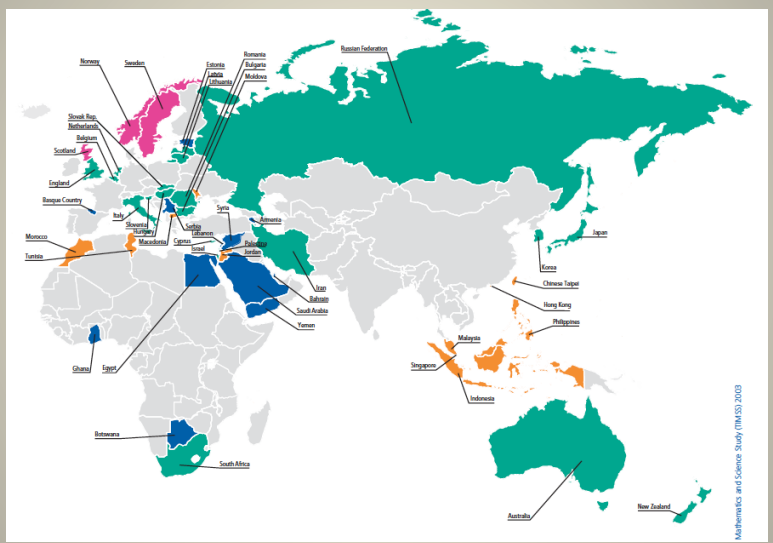
(Standards and Focal Points, NCTM)

- Problem solving means engaging in a task for which the solution is not known in advance.
- Good problems give students the chance to solidify and extend their knowledge and to stimulate new learning. Most mathematical concepts can be introduced through problems based on familiar experiences coming from students' lives or from mathematical contexts.
- Students need to develop a range of strategies for solving problems, such as using diagrams, looking for patterns, or trying special values or cases.

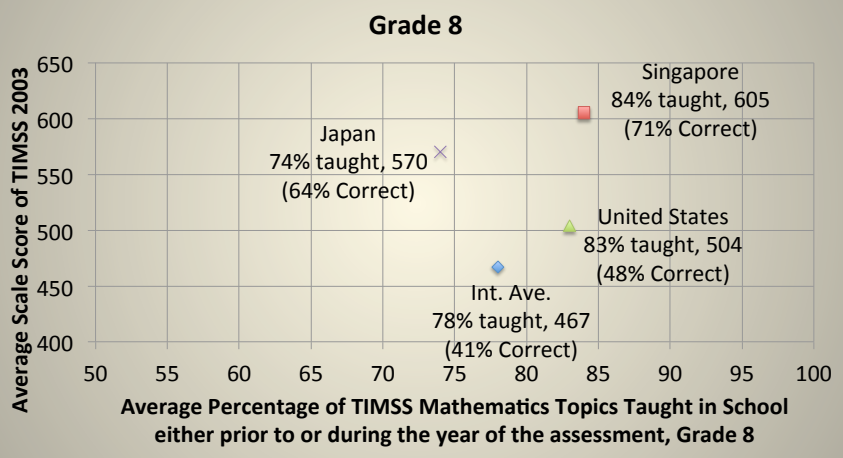
Key publications have influenced how problem solving is used in Japanese mathematics classrooms

- Polya's How to Solve It (Polya, 1945)
- An Agenda for Action: Recommendations for School Mathematics of the 1980s (NCTM 1980)
- Teaching Problem Solving: What, why & how (Charles & Lester, 1982) was translated into Japanese in 1983
- NCTM Curriculum and evaluation standards for school mathematics (NCTM 1989)
- NCTM's Principles and Standards for School Mathematics NCTM 2000)

TIMSS 2003



Average Percentage of TIMSS Mathematics Topics Taught in School and the Achievement (Average Scale Score) of the TIMSS 2003



Source TIMSS 2003 International Mathematics Report
Grade 8: Exhibit 5.7 (p.192), Exhibit C. 1 (p.400)
Grade 4: Exhibit 5.7 (p.193), Exhibit C. 1 (p.402)

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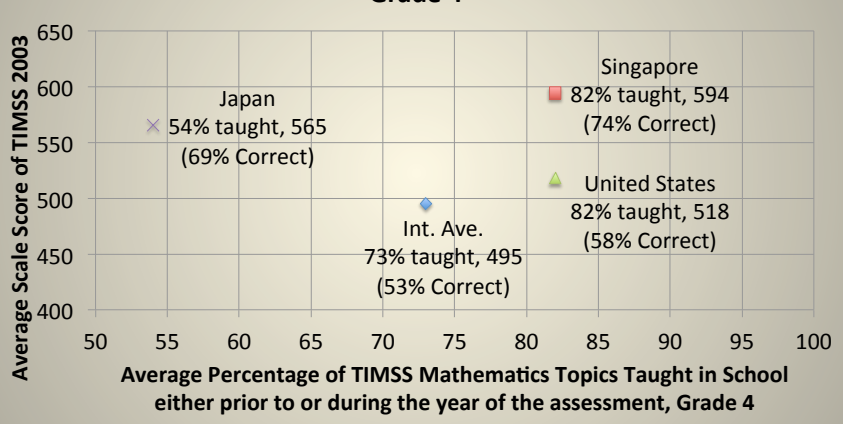

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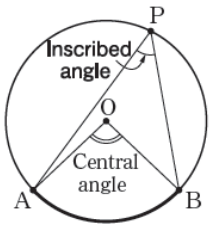
From where should we take the picture?

The photographs below were taken from different points so that both edges of the blackboard would just fit into the pictures.

In the same way, we want to take a picture so that both edges of the blackboard just fit into the picture. If we do not zoom in or out with the camera, are there many points from where we can take such pictures?

Inscribed angles

If point P is any point on circle O except for those on $\overset{\frown}{AB}$, $\angle APB$ is called an **inscribed angle** that corresponds to $\overset{\frown}{AB}$. Also, $\overset{\frown}{AB}$ is said to be an arc that corresponds to $\angle APB$. In circle O, there are many inscribed angles $\angle APB$ on $\overset{\frown}{AB}$.



Let's think about a way to prove that the measure of inscribed angles of a given arc is constant.

Q The figures below show $\overset{\frown}{AB}$ and 2 of its corresponding inscribed angles $\angle APB$ as the location of point P is changed. In Figures (a) and (b), those angles are drawn separately.
As point P is moved, what stays constant in $\triangle OPA$ and $\triangle OPB$?

Proof Draw diameter PC, and let $\angle OPA = \angle a$ and $\angle OPB = \angle b$.

Since $OP = OA$, $\angle OAP = \angle a$.

Since $\angle AOC$ is an exterior angle of $\triangle OAP$,

$$\angle AOC = \angle OPA + \angle OAP = 2\angle a$$

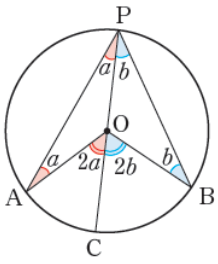
Similarly, $\angle BOC = 2\angle b$.

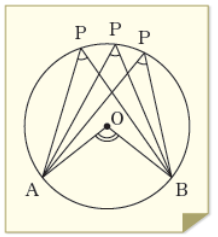
Therefore, $\angle AOB = 2(\angle a + \angle b)$

Since $\angle APB = \angle a + \angle b$,

$$\angle APB = \frac{1}{2}\angle AOB$$

Since the measure of the central angle $\angle AOB$ corresponding to $\overset{\frown}{AB}$ is fixed, the measure of $\angle APB$ is constant.

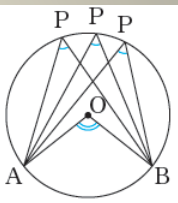




From what we have investigated, we can state the following theorem.

The Inscribed Angle Theorem

Theorem The measure of the inscribed angle corresponding to an arc is constant and it is half the measure of the central angle corresponding to the arc.



$$\angle APB = \frac{1}{2}\angle AOB$$

Mathematics International, English translation of Japanese math textbooks for Grade 9 (15 years old) pp. 168-172
http://www.globaledresources.com/

Three Levels of Teaching

Japanese mathematics educators and teachers identify three levels of expertise of mathematics teaching:

- Level 1: The teacher can tell students the important basic ideas of mathematics such as facts, concepts, and procedures.
- Level 2: The teacher can explain the meanings and reasons of the important basic ideas of mathematics in order for students to understand them.
- Level 3: The teacher can provide students with opportunities to understand these basic ideas, and support their learning so that the students become independent learners.

(Sugiyama, Y. 2008, Trans. Takahashi, A., 2011a)

Beliefs about teaching and learning mathematics

Reprint from *Principles to Actions p.11* (NCTM, 2014)

Unproductive beliefs	Productive beliefs
Mathematics learning should focus on practicing procedures and memorizing basic number combinations.	Mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse.
Students need only to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems.	All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures.
Students can learn to apply mathematics only after they have mastered the basic skills.	Students can learn mathematics through exploring and solving contextual and mathematical problems.
The role of the teacher is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve mathematics problems.	The role of the teacher is to engage students in tasks that promote reasoning and problem solving and facilitate discourse that moves students toward shared understanding of mathematics.
The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests.	The role of the student is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others.
An effective teacher makes the mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused.	An effective teacher provides students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics.

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Obstacles

(Principles to Actions, p.9)

- Dominant cultural beliefs about the teaching and learning of mathematics continue to be obstacles to consistent implementation of effective teaching and learning in mathematics classrooms (Handal 2003; Philipp 2007).
- Many parents and educators believe that students should be taught as they were taught, through memorizing facts, formulas, and procedures and then practicing skills over and over again (e.g., Sam and Ernest 2000).
- This view perpetuates the traditional lesson paradigm that features review, demonstration, and practice and is still pervasive in many classrooms (Banilower et al. 2006; Weiss and Pasley 2004).

- Listening to experts during special professional development days does not translate into improved teaching. Effective teacher learning must be built into teachers' daily and weekly schedules. Schools must become the places where teachers, not just students, learn.

(Closing the Teaching Gap, 2009)

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For Level 3 teaching

- In order to develop expertise for Level 3 teaching, learning by reading, listening, and seeing may not be sufficient.

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Two Major Types of Professional Development

- Phase 1 professional development focuses on developing the knowledge for teaching mathematics,
  - through reading books and resources, listening to lectures, and watching visual resources such and video and demonstration lessons.
- Phase 2 professional development focuses on developing expertise for teaching mathematics
  - teachers should plan the lesson carefully, teach the lesson based on the lesson plan, and reflect upon the teaching and learning based on the careful observation. Japanese teachers and educators usually go through this process using Lesson Study

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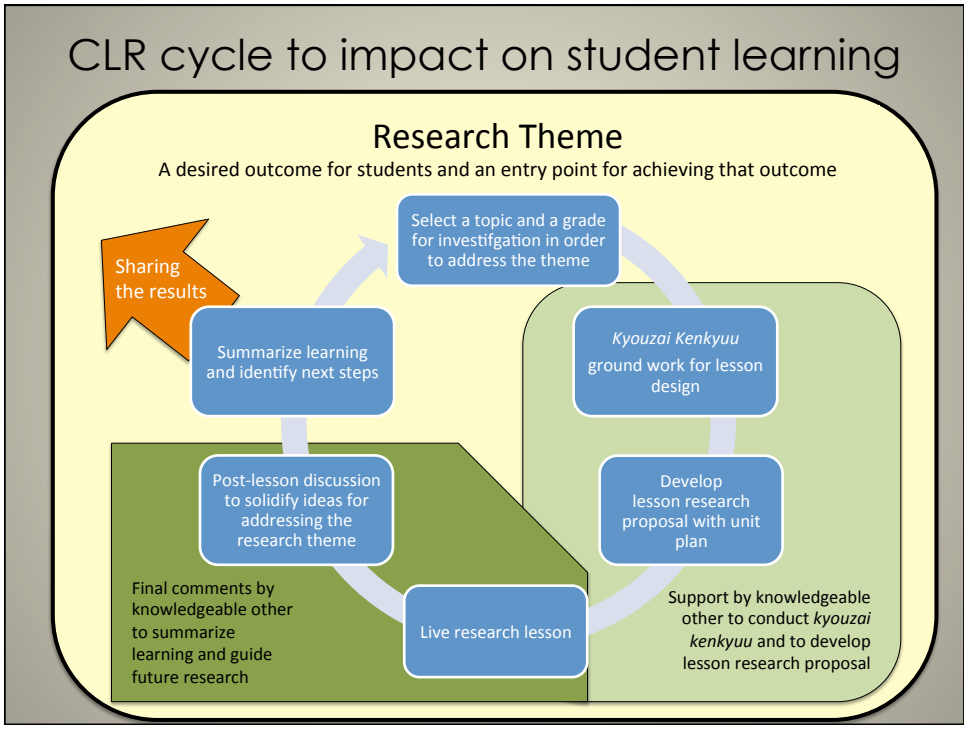
Lesson Study is not an end in itself, but a process for accomplishing specific teaching-learning goals.

Collaborative Lesson Research (CLR) for Maximizing the impact of Lesson Study

We define Collaborative Lesson Research (CLR) has having the following components:

1. A clear research purpose
2. Kyouzai kenkyuu
3. A written research proposal
4. A live research lesson and discussion
5. Knowledgeable others
6. Sharing of results


CLR cycle to impact on student learning
Research Theme
A desired outcome for students and an entry point for achieving that outcome
Select a topic and a grade for investifgation in order to address the theme
Kyouzai Kenkyuu
ground work for lesson design
Develop lesson research proposal with unit plan
Support by knowledgeable other to conduct kyouzai kenkyuu and to develop lesson research proposal
Live research lesson
Post-lesson discussion to solidify ideas for addressing the research theme
Final comments by knowledgeable other to summarize learning and guide future research
Summarize learning and identify next steps
Sharing the results