Patterns, Sequences and Graphing

A sequence is a set of numbers in a particular order.

Types of Patterns:

Linear: 3, 6, 9, 12, ….
+3 +3 +3 1^{st} Difference

In a linear sequence the first difference is constant (same number every time). This will produce a straight line when graphed. A linear sequence can also be called an arithmetic sequence.

Quadratic: 1, 6, 15, 28, 45, ….
+5 +9 +13 +17 1^{st} Difference
+4 +4 +4 2^{nd} Difference

In a quadratic sequence the second difference is constant. This will produce a curve when graphed.

Exponential: 2, 4, 8, 16, …. (multiply by 2 each time or 2^1, 2^2, 2^3, 2^4, …..

An exponential sequence is a sequence where each term is multiplied by a particular number to find the next term. It is used for exponential growth (increase) or decay (decrease) and its graph will increase or decrease very fast.

2018 Paper 1 Question 1

(a) Only one linear pattern begins with "1, 7".

Fill in the three boxes below so that the numbers form this **linear** pattern.

Linear pattern: 1, 7, $\boxed{13}$, $\boxed{19}$, $\boxed{25}$

Since this is a linear pattern, the 1^{st} difference = 7 – 1 = 6, so we add 6 to each term to get the next term.

(b) Many different quadratic patterns begin with "1, 7".

Fill in the three boxes below so that the numbers form a **quadratic** pattern.

Quadratic pattern: 1, 7, $\boxed{14}$, $\boxed{22}$, $\boxed{31}$
+6 + 7 +8 +9
+1 +1 +1

Since this is a quadratic pattern, the 1^{st} difference is not the same each time. We can choose any constant second difference. In this case we chose 1 for the 2^{nd} difference.

2017 Paper 1 Question 11

(a) In a particular **linear** sequence, the second term is 40 and the sixth term is 116.
Fill in the boxes below to show the rest of the first six terms of this sequence.

$\boxed{21}$, $\boxed{40}$, $\boxed{59}$, $\boxed{78}$, $\boxed{97}$, $\boxed{116}$

Since this is a linear sequence the 1st difference is constant.

To find the 1st difference we subtract 40 from 116 and divide by 4 since there are 4 differences between them.

$$116 - 40 = 76$$
$$\frac{76}{4} = 19$$
$$40 - 19 = 21$$
$$40 + 19 = 59 \text{, etc.}$$

(b) Orla is asked to write down a **quadratic** sequence. She writes down the following:

5, 6, 9, 14, 22, 30, 41

+1 +3 +5 +8 +8 +11 1st Difference

+2 +2 +2 +1 +3 2nd Difference

Exactly **one** of the terms in Orla's sequence is incorrect.

Write down the correct quadratic sequence in the spaces below.
You may only change **one** of the terms in Orla's sequence.

Since this is a quadratic sequence the 2nd difference is constant (in this case it is +2).

We first find the 1st difference and then the 2nd difference. We see that the 2nd difference changes between 14 and 22 so that is the wrong number.

Answer: 5, 6, 9, 14, 21, 30, 41

Finding the nth term of Quadratic Sequences

The nth term of a quadratic sequence will always be of the form:

$$\boxed{T_n = an^2 + bn + c}$$

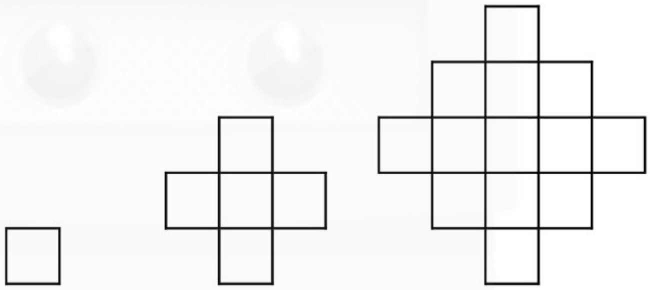
We use simultaneous equations to find the values of b and c.

In a quadratic sequence, the coefficient of n^2 in the nth term is half the 2^{nd} difference.

2015 Sample Paper 2 Question 8

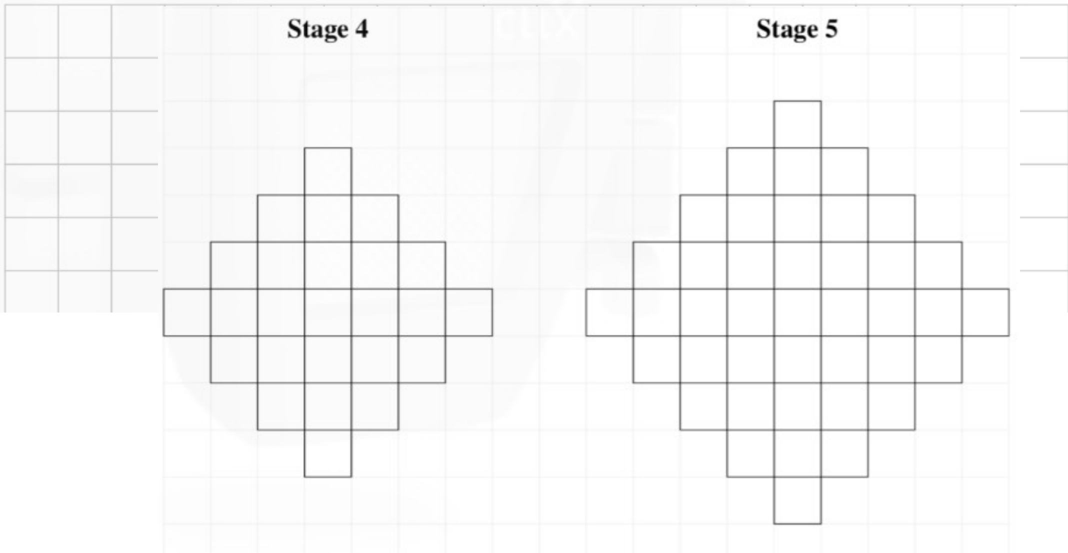
The first three stages of a pattern are shown below.

Each stage of the pattern is made up of small squares.
Each small square has an area of one square unit.



Stage 1 **Stage 2** **Stage 3**

(a) Draw the next two stages of the pattern.



(b) The perimeter of Stage 1 of the pattern is 4 units.
The perimeter of Stage 2 of the pattern is 12 units.

Find a general formula for the **perimeter** of Stage n of the pattern, where $n \in \mathbb{N}$.

We'll consider the perimeters of the first five Stages:

Stage	1	2	3	4	5
Perimeter	4 units	12 units	20 units	28 units	36 units

From one stage to another, there is a difference of 8 in the perimeter. Thus, the perimeter is an arithmetic sequence with first element $a = 4$ and common difference $d = 8$. The general formula for the perimeter of Stage n is therefore

$$P_n = a + (n-1)d = 4 + 8(n-1) = 8n - 4$$

(c) Find a general formula for the **area** of Stage n of the pattern, where $n \in \mathbb{N}$.

Area Sequence = 1, 5, 13, 25, 41

+4 +8 +12 +16 1st Difference

+4 +4 +4 2nd Difference

Since the 2nd Difference is constant then it is a quadratic sequence.

We will use $T_n = an^2 + bn + c$ where $a = \frac{4}{2} = 2$, so $T_n = 2n^2 + bn + c$

Now, we know that $A_1 = 1$ and $A_2 = 5$, so

$$1 = A_1 = 2(1)^2 + b(1) + c = 2 + b + c \quad \Rightarrow \quad b + c = -1$$
$$5 = A_2 = 2(2)^2 + b(2) + c = 8 + 2b + c \quad \Rightarrow \quad 2b + c = -3$$

We will solve these equations simultaneously:

$$\begin{array}{rcl} b+c & = & -1 \\ 2b+c & = & -3 \\ \hline b & = & -2 \end{array}$$

Now, from the first of our two simultaneous equations to see that $(-2) + c = -1$ or $c = -1 + 2 = 1$. This means that our sequence has the form $A_n = 2n^2 - 2n + 1$

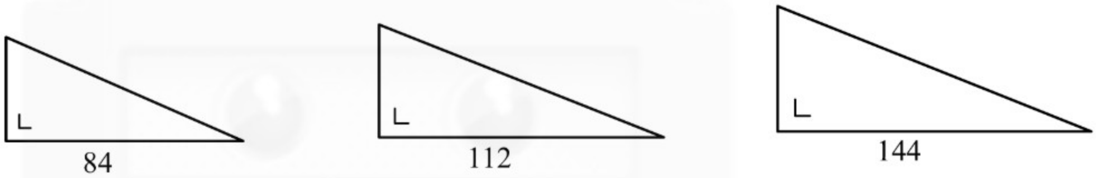
(d) What kind of sequence (linear, quadratic, exponential, or none of these) do the **areas** follow? Justify your answer.

This sequence is quadratic because the highest power in the general formula is n^2. It cannot be exponential because there is no term of the form a^n.

2015 Paper 1 Question 11

These triangles can be put in a sequence of increasing size.
The lengths of the bases of the triangles in this sequence follow a **quadratic** pattern.
Three consecutive triangles in this sequence are shown below.

(c) Use this information to find the length of the base of the next triangle in the sequence.



The length of the hypotenuse, h, of triangle x in this sequence is given by the function below, where b and c are integers.

$$h(x) = 2x^2 + bx + c$$

Also, $h(1) = 5$ and $h(2) = 13$.

(d) **(i)** Use this information to write two equations in b and c.

Equation 1:	Equation 2:
$h(1) = 2(1)^2 + b(1) + c = 5$	$h(2) = 2(2)^2 + b(2) + c = 13$
$\Rightarrow \quad 2 + b + c = 5$	$\Rightarrow \quad 8 + 2b + c = 13$
$\Rightarrow \quad b + c = 3$	$\Rightarrow \quad 2b + c = 5$

(ii) Solve these simultaneous equations to find the value of b and the value of c.

Equation $1 \times (-1)$: $\quad -b - c = -3$

Equation 2 : $\quad \underline{2b + c = 5}$

$\Rightarrow \quad b = 2 \quad \Rightarrow \quad c = 1.$

$b =$ 2 $\qquad\qquad c =$ 1

Repeating Patterns

Sometimes a pattern will repeat in blocks of a certain number of terms.

e.g. **A, B, C, D, A, B, C, D, A, B, ….** this pattern repeats every 4 terms.

We can find what the letter will be for any term, like the 67^{th} term, by finding how many blocks of 4 are in 67 and then how many letters are left over:

$67 \div 4 = 16$ and remainder 3.

Since the remainder is 3, we look for the 3^{rd} letter in the block of 4. In this case it is C.

2019 Paper 1 Question 9

Gertie writes down the following sequence, which repeats every three terms:

3, 6, 4, 3, 6, 4, 3, …

The 1st term is 3.

(a) **(i)** Write down the value of the 12th term.

Answer: 4

3, 6, 4, 3, 6, 4, 3, 6, 4, 3, 6, 4

(ii) Work out the value of the 100th term in this sequence.

Answer: 3

(ii) $100 \div 3 = 33\ Rem\ 1$ so 100^{th} term will be the same as Term 1

100^{th} term = 3

(b) Describe how to find the value of the nth term in the sequence, where $n \in \mathbb{N}$, **without** listing all the terms from the 1st to the nth.

Divide n by 3.

If remainder = 0 then $T_n = T_3$

If remainder = 1 then $T_n = T_1$

If remainder = 2 then $T_n = T_2$

Gertie made her sequence $3, 6, 4, 3, 6, 4, 3, \ldots$ by picking 3 as the 1st term, and then using **this rule**:

> If a term is **odd**, multiply it by 2 to get the next term.
>
> If a term is **even**, add 2 to it and half your answer to get the next term.

For example, 3 is odd, so the next term is 2×3, which is 6.

6 is even, so the next term is $\frac{1}{2} \times (6+2)$, which is 4.

(c) A different sequence follows the **same rule**, but has **8 as the 1st term**.
Work out the next four terms of this sequence.

Answer: $\boxed{8}$, $\boxed{}$, $\boxed{}$, $\boxed{}$, $\boxed{}$

Working out:

2nd term $= \frac{1}{2} \times (8+2) =$

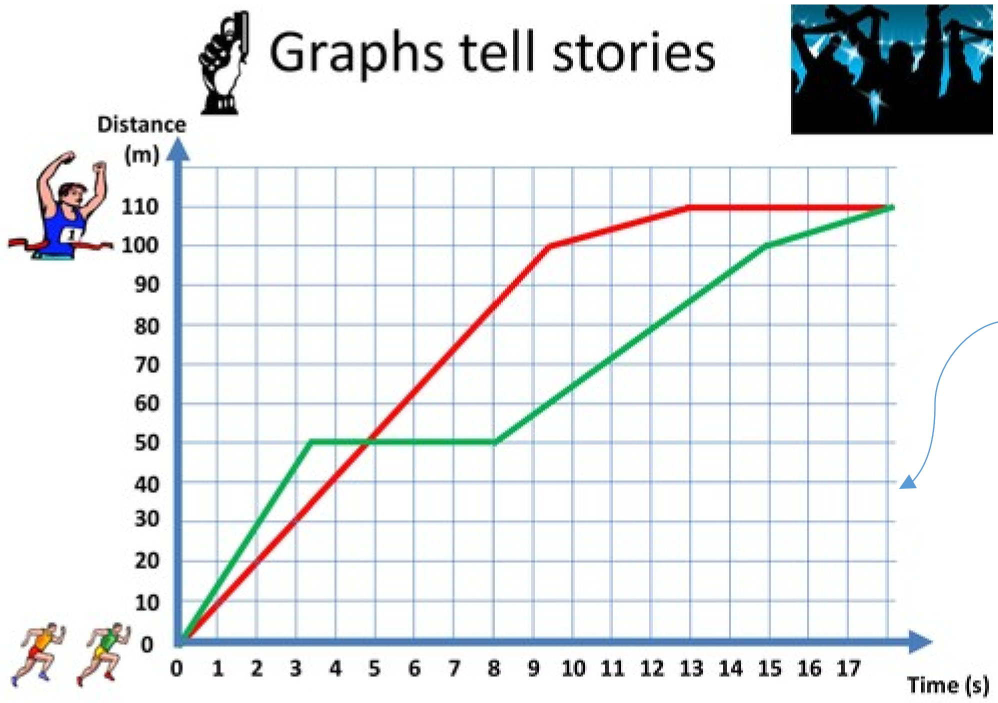
(d) Ahmed takes **2 as his 1st term**, and makes a sequence using the **same rule**.

State what is unusual about Ahmed's sequence.
It might be helpful to work out some of the terms of his sequence.

His sequence will remain constant at 2 *or equivalent*

Graphing

Graphs are very important in Maths as they are a way of representing the relationship between two variables (a variable is a quantity that can change).



Here we see the relationship between time and distance when we run.

It is important that we are always aware of each variable that is changing on the horizontal axis and on the vertical axis. We can see in the graph above the relationship between time and distance travelled.

It is also useful to make a connection between slope and the variables on each axis. You may remember that:

$$Slope = \frac{Rise}{Run} \vee Slope = \frac{Vertical\,Change}{Horizontal\,Change}$$

For the graph above we know that:

vertical change = distance (m) and

horizontal change = time (s)

so in this case: $$Slope = \frac{Vertical\,Change}{Horizontal\,Change} = \frac{Distance}{Time}$$

which we know is speed (m/s).

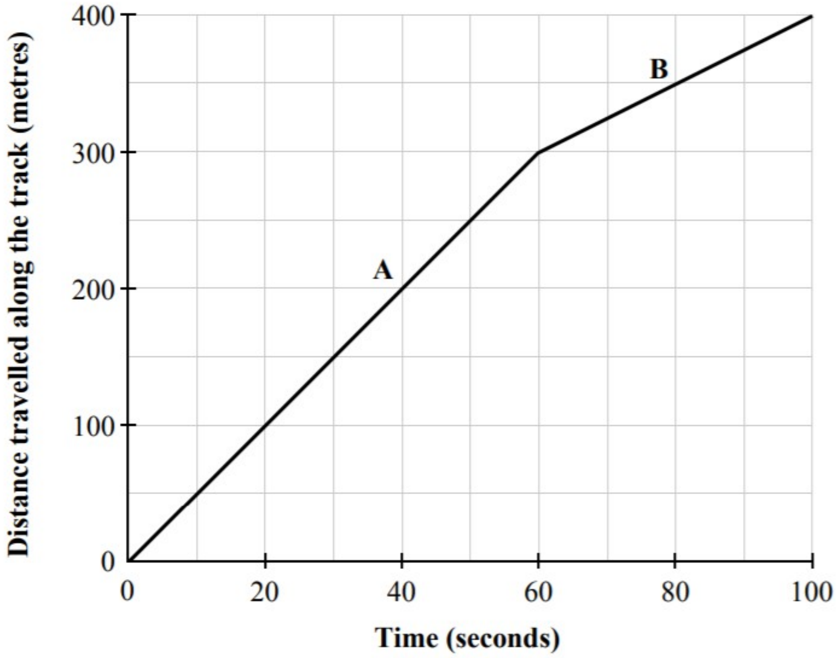
Analysing the slope:

The slope of a graph gives us information about the graph too.

- In the above case the steeper the line then the greater speed.
- When the line is horizontal then there is no change in distance so the speed = 0 m/s.

2016 Paper 1 Question 5

(a) The graph below shows the distance travelled along a track by Ann over the course of a race. The graph is in two sections, labelled **A** and **B**.



(i) Show that Ann's speed in section **A** is 5 metres per second.

Use: Slope = $\frac{\text{Distance}}{\text{Time}}$

(ii) Find Ann's speed in section **B**, in metres per second.

(b) **Table 1** shows graphs of the **distance** travelled along the track by Bill, Claire, and Dee during the same race. Each person's name is written next to their graph.

Table 2 shows graphs of the **speed** of Bill, Claire, and Dee during the race.
Complete Table 2, by writing the correct name next to each graph.

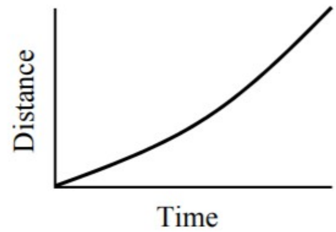
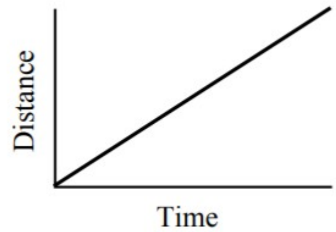
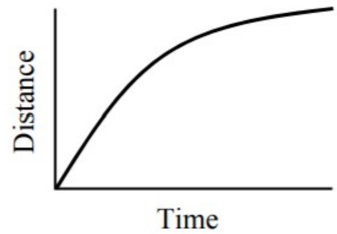
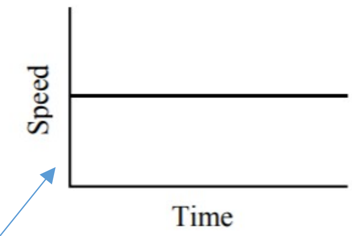
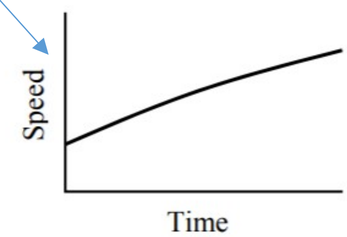
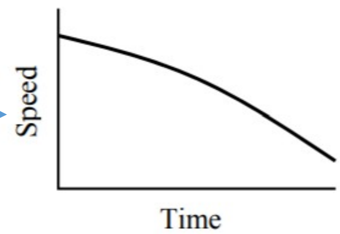
Table 1	
Distance travelled during the race	**Name**
	Bill
	Claire
	Dee

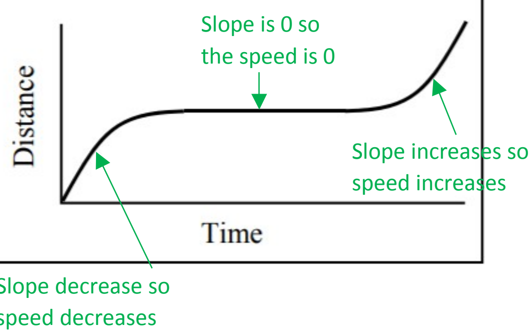
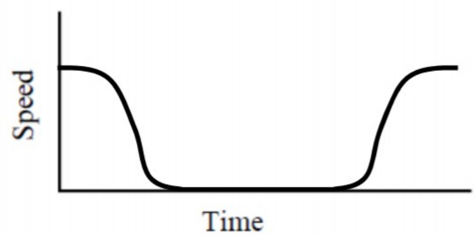
Table 2	
Speed during the race	**Name**
	Claire
	Bill
	Dee

Bill: The slope is increasing so the speed is increasing.

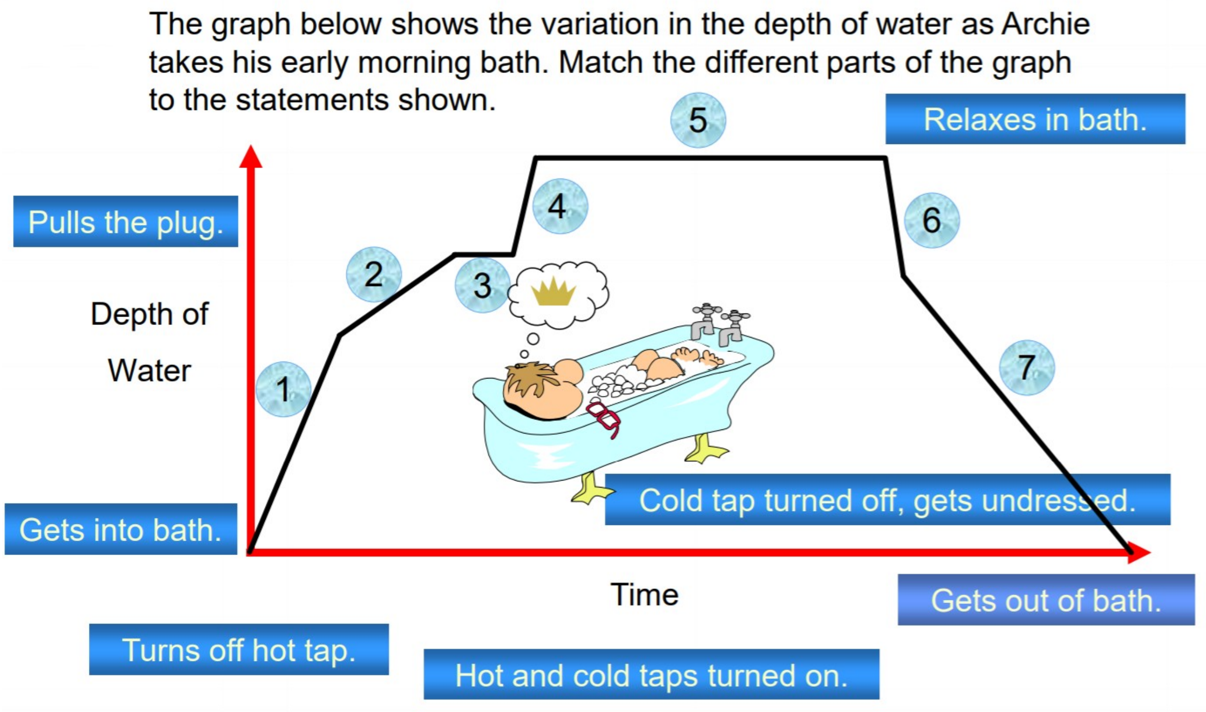
Claire: The slope doesn't change so the speed is constant.

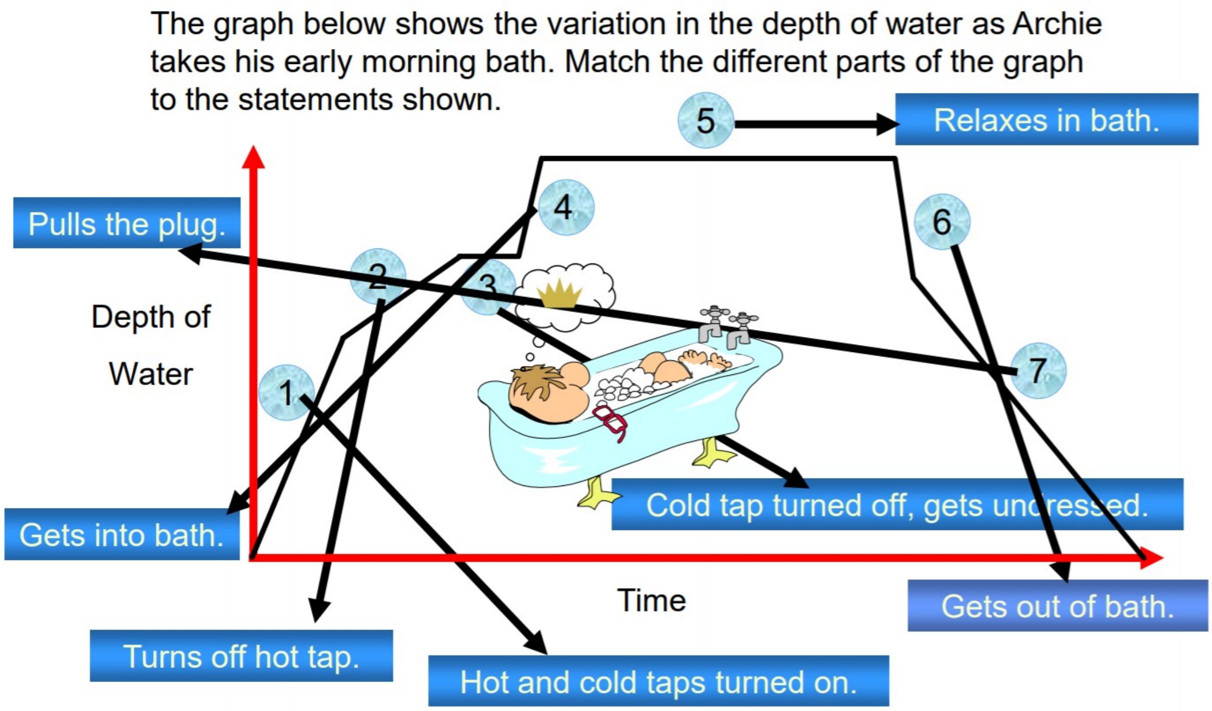
Dee: The slope begins quite steep (large speed) and then the slope decreases (speed decreases).

(c) The graph below shows the distance Erik travelled along the track during the same race.
Sketch the graph of Erik's **speed** during the race on the axes below.

Distance travelled by Erik during the race	Erik's **speed** during the race
	

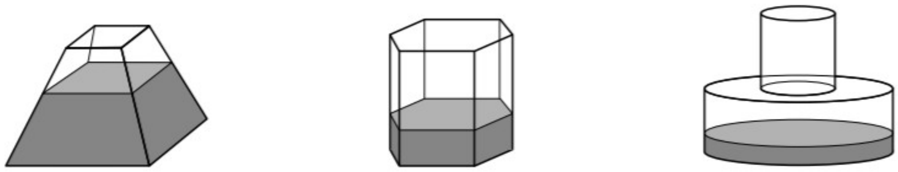
Graphs of Real Life Situations





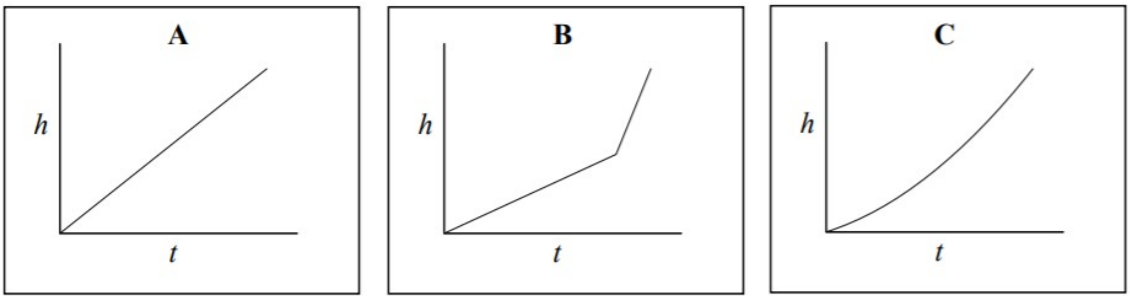
2014 Paper 1 Question 6

Below are three containers, labelled **1**, **2**, and **3**.
Water is poured into each container at a constant rate, until it is full.



1 **2** **3**

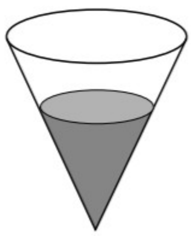
The three graphs, **A**, **B**, and **C**, show the height of the water, h, in the containers after time t.

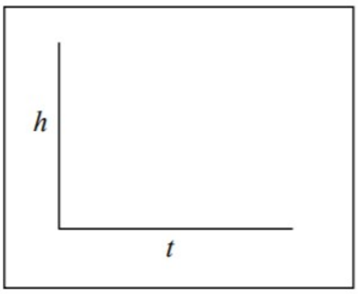


(a) Write **A**, **B**, and **C** in the table below to match each container to its corresponding graph.

Container	1	2	3
Graph			

(b) Another container is shown below. Water is also poured into this container at a constant rate until it is full. Sketch the graph you would expect to get when plotting height (h) against time (t) for this container.

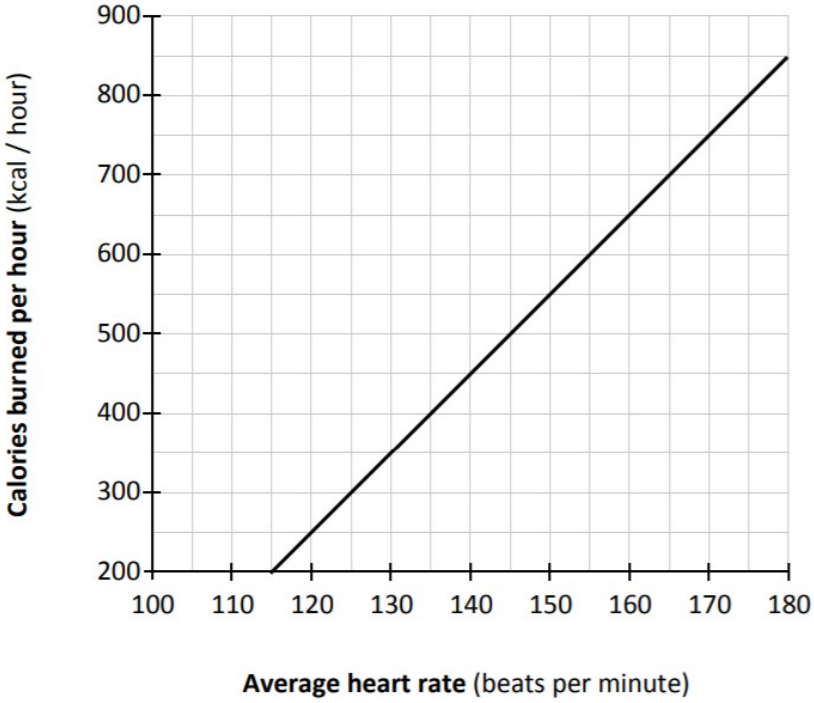




2018 Paper 1 Question 2

(a) Phil is an athlete. The graph below shows the number of calories (in kcal) she burns per hour, depending on her average heart rate.

Note: the axes do not start at $(0, 0)$.



(i) Use the graph to estimate how many calories Phil would burn in one hour if she had an average heart rate of **170 beats per minute**.

Calories burnt in one hour = $\boxed{750}$ kcal

(ii) Use the graph to work out Phil's average heart rate when she burns **300 kcal** in **30 minutes**.

Average heart rate = $\boxed{155}$ beats per minute

300 kcal in 30 mins

means 600 kcal in 1 hour

∴ 155 [beats per minute]

(b) Phil runs a series of stages. In each stage she runs a slow run of 60 m, and then a sprint. In each stage after stage 1, she sprints 50 m more than she did in the previous stage.

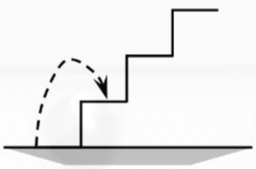
Complete the table below, showing the distance that she runs slowly and the distance that she sprints in each stage, as well as the total distance per stage.
Give the values in the last row in terms of n, where appropriate.

Stage	Slow run (metres)	Sprint (metres)	Total distance (metres)
1	60	50	110
2	60	100	160
3	60	150	
4			
5			
n			

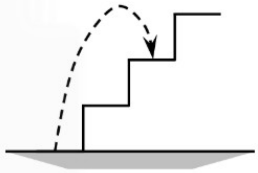
Stage	Slow	Sprint	Total
1	60	50	110
2	60	100	160
3	60	150	210
4	60	200	260
5	60	250	310
n	60	$50n$	$60 + 50n$

2015 Paper 1 Question 14

A boxer runs up stairs as part of her training.
She can go up 1 step or 2 steps with each stride, as shown.



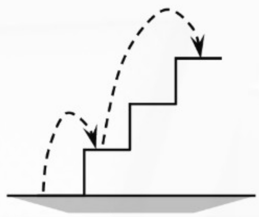
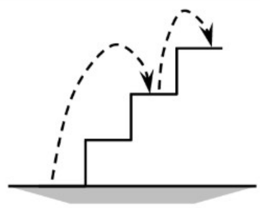
Up 1 step



Up 2 steps

The boxer wants to count how many different ways she can reach the nth step.
She calls this T_n, the nth Taylor number.

For example, she has **3** different ways to reach the 3rd step, as shown in the tables below.
So $\boldsymbol{T_3 = 3}$.

3rd step: way 1	3rd step: way 2	3rd step: way 3
Up 1 step, then 1 step, then 1 step	Up 1 step, then 2 steps	Up 2 steps, then 1 step
1 + 1 + 1	1 + 2	2 + 1
		

(a) Find the value of T_1 and T_2.

$T_1 = 1$ [way] [1 step]	$T_2 = 2$ [ways] [1 step + 1 step or 2 steps]

$T_1 =$ 1 $T_2 =$ 2

(b) **List** all the different ways that she can reach the 4th step; one way is already done for you.
Hence **write down** the value of T_4.

Different ways to reach the 4th step: **1 + 1 + 1 + 1**

1 + 1 + 2

1 + 2 + 1

2 + 1 + 1

2 + 2 [steps]

$\Rightarrow$ $T_4 = 5.$ [ways]

Answer: $T_4 =$ 5

`

Some of the ways to reach the nth step start by going up **1 step**; others start by going up **2 steps**.

(c) **(i)** **List** the different ways that she can reach the 5th step, if she starts by going up **1 step**.

1 + 1 + 1 + 1 + 1
1 + 1 + 1 + 2
1 + 1 + 2 + 1
1 + 2 + 1 + 1
1 + 2 + 2 [steps]

[5 ways]

(ii) **List** the different ways that she can reach the 5th step, if she starts by going up **2 steps**.

2 + 1 + 1 + 1
2 + 1 + 2
2 + 2 + 1 [steps]

[3 ways]

(d) **Explain** why $T_{100} = T_{99} + T_{98}$.

To get to the 100th step, you must start by going up either 1 step or 2 steps.

If you start by going up 1 step, there are T_{99} ways to finish.

If you start by going up 2 steps, there are T_{98} ways to finish.

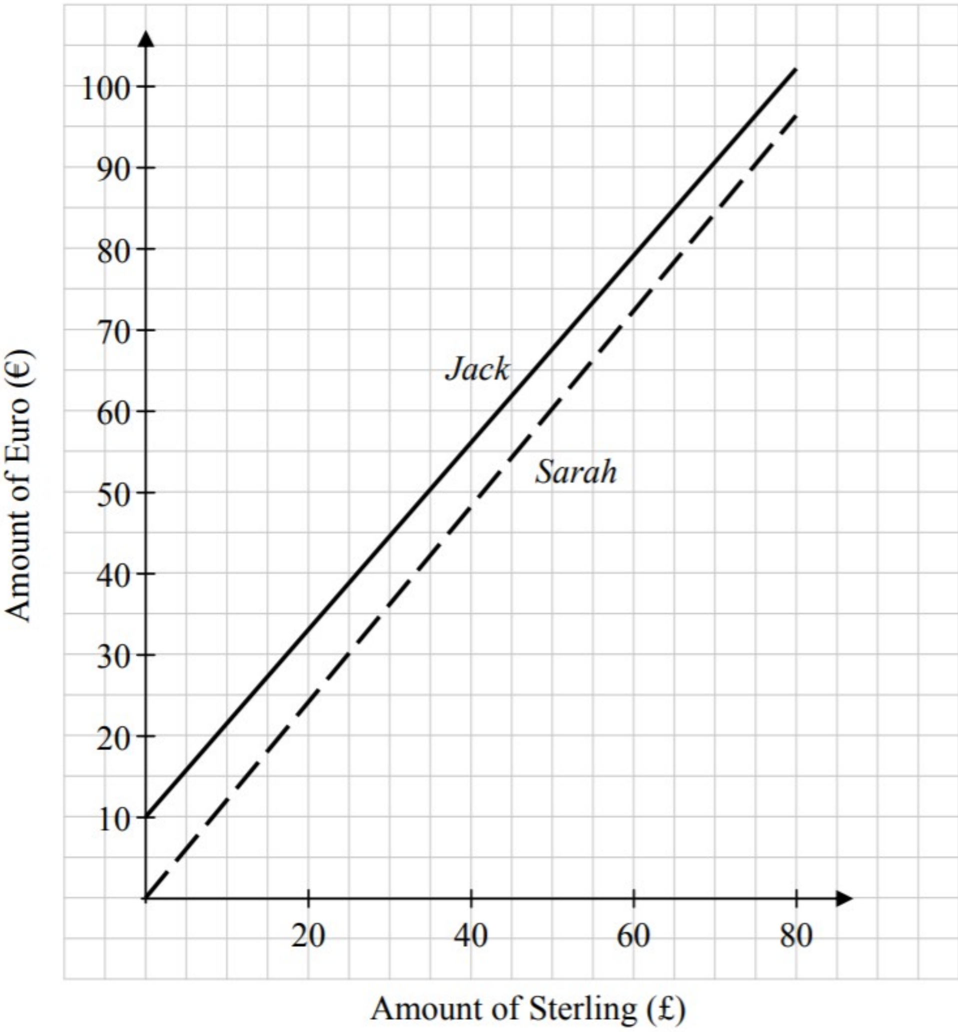
2014 Paper 1 Question 9

Jack and Sarah are going on a school tour to England. They investigate how much different amounts of sterling (£) will cost them in euro (€). They each go to a different bank.

Their results are shown in the table below.

Amount of sterling (£)	Cost in euro (€) for Jack	Cost in euro (€) for Sarah
20	33	24
40	56	48
60	79	72
80	102	96

(i) On the grid below, draw graphs to show how much the sterling will cost Jack and Sarah, for up to £80.



(ii) Using the table, or your graph, find the slope (rate of change) of Jack's graph. Explain what this value means. Refer to both euro and sterling in your explanation.

Slope: $\dfrac{56-33}{40-20}=\dfrac{23}{20}$, *or* $1{\cdot}15$.

Explanation: Each extra £1 costs Jack an extra €$1{\cdot}15$.

(iii) Write down a formula to represent what Jack must pay, in euro, for any given amount of sterling. State clearly the meaning of any letters you use in your formula.

Since after €10 every amount in euro will be 1.15 × (Sterling amount)

$e=1{\cdot}15s+10$, where s is the amount, in sterling, and e is the amount, in euro.

(iv) Write down a formula to represent what Sarah must pay, in euro, for any given amount of sterling. State clearly the meaning of any letters you use in your formula.

$e=1{\cdot}15s+10$, where s is the amount, in sterling, and e is the amount, in euro.

(v) Using your formulas from **(iii)** and **(iv)**, or otherwise, find the amount of sterling Jack and Sarah could buy that would cost them the same amount each in euro.

Using formulas:

$e=1{\cdot}15s+10$ and $e=1{\cdot}2s$, so $1{\cdot}15s+10=1{\cdot}2s$,

i.e. $s=200$ and $e=240$.

Amount of sterling: £200.

From table:

Each time the amount of sterling goes up by 20, the difference between the costs decreases by €1.

This difference is €9 for £20.

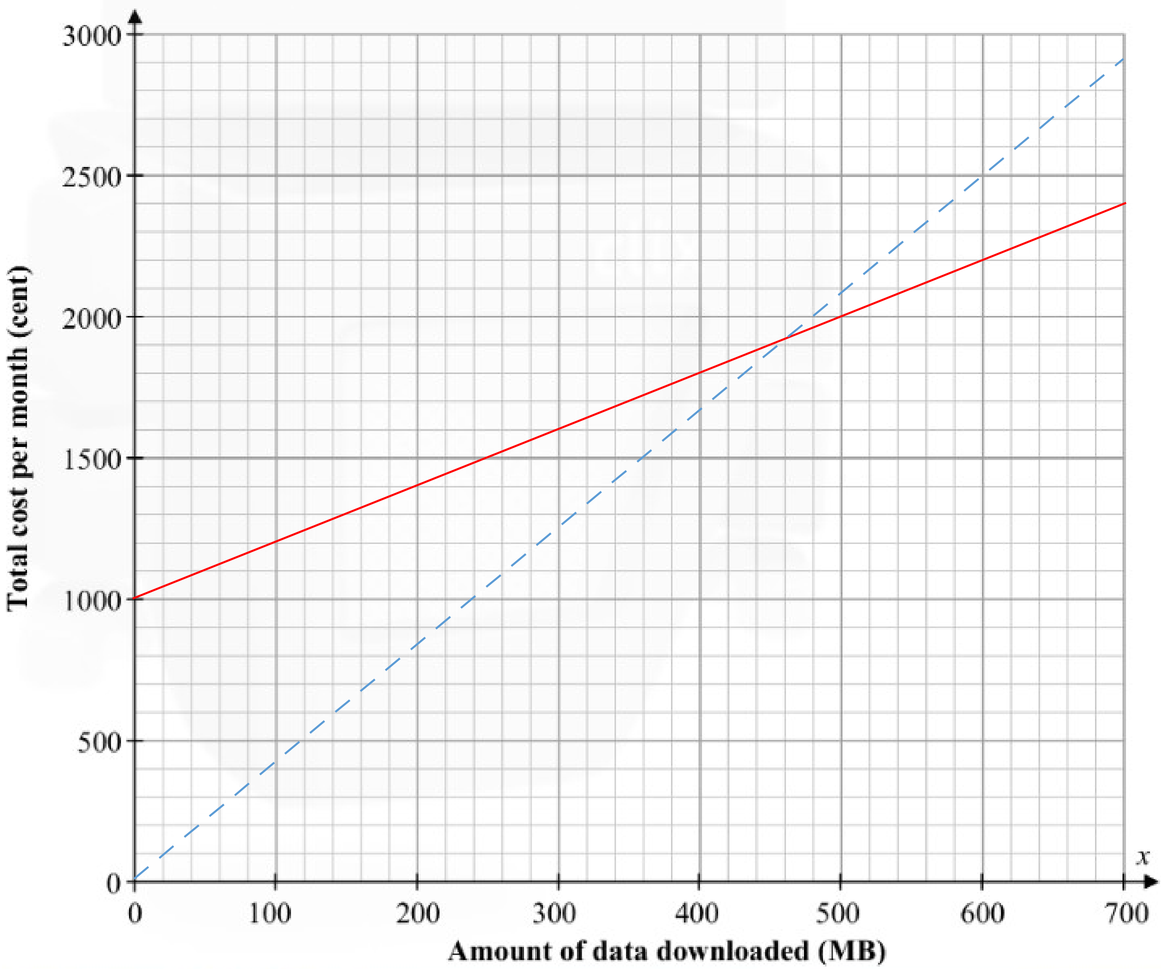
So after 9 increases, i.e. increase of $9\times 20=£180$, the costs are the same, i.e. for £200.

2015 Question 6

Two mobile phone companies, *Cellulon* and *Mobil*, offer price plans for mobile internet access. A formula, in x, for the total cost per month for each company is shown in the table below. x is the number of MB of data downloaded per month.

Phone company	Total cost per month (cent)
Cellulon	$c(x) = 4x$
Mobil	$m(x) = 1000 + 2x$

(a) **Draw** the graphs of $c(x)$ and $m(x)$ on the co-ordinate grid below to show the total cost per month for each phone company, for $0 \le x \le 700$. **Label** each graph clearly.



To draw a line we need to two points. In this case we find the start and end point for each.

(b) Which company charges **no** fixed monthly fee?
Justify your answer, with reference to the relevant **formula** or **graph**.

Answer: *Cellulon*

Justification: $c(0) = 0$, so no cost if no data used. **(formula)**

OR

The line goes through (0, 0), so no cost if no data used. **(graph)**

When the data used is zero then the cost is zero.

(c) Write down the **point of intersection** of the two graphs.

(500, 2000)

The point of intersection is where the two lines cross.

Fergus wants to buy a mobile phone from one of these two companies, and wants his mobile internet bill to be as low as possible.

(d) **Explain** how your answer to part **(c)** would help Fergus choose between *Cellulon* and *Mobil*.

If the data is less than 500 MB per month, *Cellulon* is cheaper.

If the data is more than 500 MB per month, *Mobil* is cheaper.